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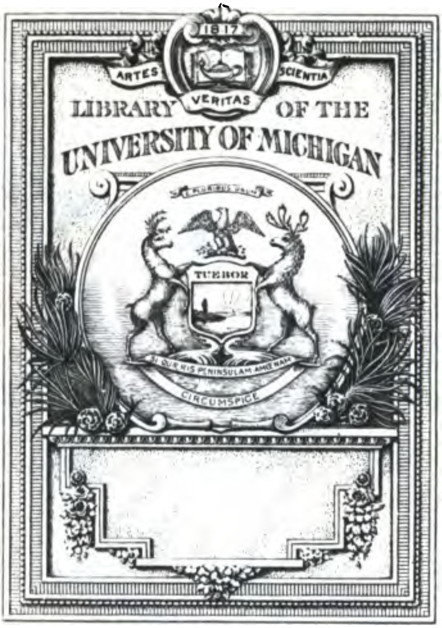
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# ELEMENTS

OF

# GEOMETRY,

WITH

## PRACTICAL APPLICATIONS.

DESIGNED FOR BEGINNERS.

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BY GEORGE R. PERKINS, A.M.

Professor of Mathematics in the New-York State Normal School; Author of
"Elementary Arithmetic," "Higher Arithmetic," "Elements of
Algebra," "Treatise of Algebra," &c. &c.

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# PREFACE.

THERE are two methods of investigating the principles of Geometry. The only method known to the ancients was independent of the aid of Algebra. This method has been so completely developed by EUCLID, as to leave little room for improvement. It is true, modern writers have arrived at many of his conclusions by more simple and concise methods; but, in so doing, they have, in most instances, sacrificed that rigor of logical demonstration, which so justly constitutes the great merit of his writings.

While but little room is left for improving on the model of EUCLID, the modern geometer, by bringing to his aid the principles of Algebra, has greatly enriched the geometry of the ancients, by the discovery of many beautiful relations of magnitudes, which probably would never have been brought to light by the old method.

In this work, which is after the model of EUCLID, we have not strictly copied any one author, but have endeavored to select from all the sources within our reach, such parts as we deemed best adapted to our wants. In the solid geometry, or geometry of three dimensions, we have made free use of PETER BARLOW's arrangement, as given in the *Encyclopædia Metropolitana;* which, indeed, is but a slight modification of LEGENDRE's method.

We have found, from experience in teaching, that, as a general thing, beginners in the study of geomety consider it as *a dry, uninteresting science.* They have but little difficulty in following the demonstration, and arriving at a full conviction of its truth; but they ask, What if the proposition is true? What use can be made of it?

Now, to meet these difficulties, we have all along in the body of the work added, in a smaller sized type, such remarks, suggestions and practical applications as we have found from experience to interest the pupil. Our object has not been to multiply these practical applications, but merely to give in their proper places a few of the more simple cases, such as would naturally suggest themselves to the mind of a successful teacher. A few examples, given in this way, will excite in the pupil a desire to invent for himself still further applications, thus keeping up a lively interest in the study of this most important branch of education.

The arrangement of the work is such as to make the text, which is given in the larger sized type, wholly independent of the explanatory matter in small type. The course is, therefore, complete with the omission of the practical portion.

In an appendix, we have given the solution of a few geometrical problems by the aid of algebra; thus showing the facility with which many difficult cases are made to yield, under the influence of the analytical method of investigation. We have also taken this opportunity to exhibit some beautiful and interesting theorems, by translating the results of algebraical deductions into the language of geometry.

GEORGE R. PERKINS.

UTICA, September, 1847.

# ELEMENTS OF GEOMETRY,

WITH

PRACTICAL APPLICATIONS.

# FIRST BOOK.

## THE PRINCIPLES.

GEOMETRY is the science of extension.

It considers the extent of distance, extent of surface, and the extent of capacity or solid content.

The name *geometry* is derived from two Greek words, signifying *land* and *to measure.*

(ART. 1.) Egypt is supposed to have been the birthplace of this beautiful and exact science, where the annual inundations of the Nile rendered it of peculiar value to the inhabitants as a means of ascertaining their effaced boundaries. At the present time it embraces the measurement of the earth and of the heavens. Its principles are applicable to magnitudes of all kinds. There is scarcely any mechanical art which does not receive great assistance from Geometry.

### DEFINITIONS OF MAGNITUDES.

I. A *solid* or *body* is a magnitude having three dimensions: length, breadth, and thickness.

II. A *surface* is the limit or boundary of a solid, having two dimensions : length, and breadth.

III. A *line* is the limit or boundary of a surface, having only one dimension : length.

IV. A *point* is not a magnitude. It has no dimension in any direction, but simply position. Hence, the extremities of lines are points. Also, the place of intersection of two lines is a point.

(2.) The common notion of a point is derived from the extremity of some slender body, such as the end of a common sewing needle. This being perceptible to the senses, is a ***physical point***, and not a ***mathematical point;*** for, by the definition, a point has no magnitude.

V. A *straight line* is the shortest distance between two points.

(3.) Among the infinite number of lines which can be imagined, having different degrees of flexure, one only corresponds with the straight line, namely, the one which has no flexure.

The outlines of the different objects of nature are, in general, presented to us in the form of curved lines, some of which are very graceful and pleasing to the eye.

(4.) In accordance with the above definition, if a fine flexible string be stretched between its two extremities, it will assume, nearly, the direction of a straight line. Owing to the weight of the string, it will necessarily be bent downwards. If, however, we could suppose the string devoid of weight, it would then produce a ***straight physical line***, which will approach more nearly to the ***mathematical line*** as the size of the string is diminished.

(5.) All the lines which we form upon paper or upon the blackboard, for the purpose of illustrating the principles of Geometry, are physical lines. Indeed, it is impossible to form a mathematical line, but we may, however, conceive of such lines, and this we must always do in our geometrical reasoning; and for the want of a better method, we use the physical lines as representatives of the mathematical lines which we wish to consider.

(6.) In ornamental gardening, the sides of walks, the rows of plants, shrubs and trees, etc. are determined by stretching a flexible cord between their extremities.

In carpentry and other arts, straight lines are formed upon plane surfaces by stretching upon the surface a flexible cord, previously rubbed over with chalk. The middle portion of the cord is then raised, and allowed to recoil by its elasticity, thus leaving upon the surface a chalked line.

(7.) Another definition of a straight line is as follows: When a line is such, that the eye being placed near one extremity so as to cause it to conceal the other extremity, it shall, at the same time, hide from view all other portions of the line; then such line is called a ***straight line.***

This definition is due to PLATO. A practical application of this definition is used by artisans, in bringing the eye to range along the direction of the line under consideration, technically called ***sighting.***

VI. Every line which is not a straight line is called a *curved line.* When we hereafter speak of a line, unless otherwise expressed, we shall mean a straight line.

Thus, AB and CD are straight lines; GH and KL are curved lines. The extremities of these lines, as well as their intersections F and M, are points.

VII. A *plane surface,* or simply a *plane,* is a surface, in which, if two points be taken at pleasure, and connected by a straight line, that line will be wholly in the surface.

(8.) A practical test, in accordance with this definition, is employed by artisans to determine whether a surface is plane. They take a rod or rule whose edge is straight, and apply it in various directions upon the surface under consideration, observing whether the edge of the rule, or as it is technically called the ***straighte-dge***, coincides in all positions with the surface; if so, the surface is a plane.

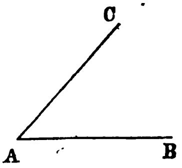
(9.) The practical miller, when he wishes to dress his millstones to a plane surface, rubs the straight-edge with paint, and then applies it in various directions upon the face of the stones; thus showing, by the transfer of the paint, which are the highest portions of the stone. These portions are dressed down; and the process again repeated, until the face of the stone has been brought as near a plane surface as may be deemed necessary.

The action of the carpenter's plane is founded upon the same principle.

VIII. Every surface which is not plane, is called a *curved surface*.

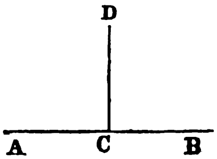
(10.) The plane surface may be regarded as one particular kind of surface out of the infinite varieties which can be imagined. To the eye, many of the curved surfaces are far more graceful and pleasing than the plane.

IX. When two straight lines AB, AC, meet each other, the space included between the lines is called an *angle*. The point of intersection A, is the *vertex* of the angle; and the lines AB, AC are the *sides* of the angle. Perhaps it would be better to define an angle as the opening between two lines which meet.

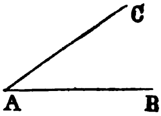


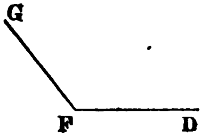
An angle is sometimes referred to by simply naming the letter at its vertex, as the angle A; but usually by naming the three letters, as the angle BAC, or CAB, observing to place the letter, at the vertex, in the middle.

X. When a straight line AB is met by another straight line DC, so as to make the adjacent angles ACD, BCD equal to each other, each is called a *right angle.* The line DC, thus meeting the line AB, is said to be perpendicular to AB.

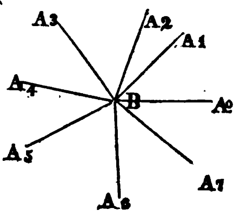


XI. Every angle BAC which is less than a right angle, is called an *acute angle;* and every angle DFG which is greater than a right angle, is called an *obtuse angle.*





(11.) If we suppose the extremity B of the line $A_0B$ to be fixed, while the line revolves in the same plane about B, so as to take the successive positions $A_1B$, $A_2B$, $A_3B$, $A_4B$, &c., until it has made a complete revolution and returned to its first position, then will this complete revolution have caused the line $A_0B$ to pass over an angular magnitude equal to four right angles. It is obvious that this angular magnitude has no dependence upon the length of the revolving line.

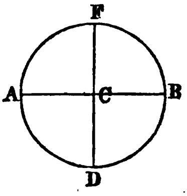


Angular magnitude is expressed numerically by supposing the whole space to be divided into 360 equal portions called *degrees*, so that 90 degrees will be the measure of a right angle. The degree is divided into 60 equal portions called *minutes*, and the minute into 60 *seconds*, and so on in sexagesimal divisions. The French mathematicians have thought it more convenient to divide the whole angular space into 400 degrees, and each degree into 100 minutes, each minute into 100 seconds, and so on in the centesimal division. In this division, the right angle would consist of 100 degrees. No doubt the French division is more simple than the sexagesimal division; but in many geometrical as well as physical inquiries, it is desirable to express certain aliquot parts of four right angles in integral degrees; and, in such cases, the sexagesimal division has the preference, since 360 has more divisors than 400.

(12.) Hereafter, unless the contrary is expressed, we shall, when speaking of degrees, wish to be understood as referring to the usual division of the whole angular space into 360 degrees. Degrees are commonly expressed by placing over the number a small circle; thus, 360° signifies 360 degrees; 90°, in the same way, denotes 90 degrees. That the student may become familiar with some of the numerical denominations of angles frequently used, we have given at one point of view the following:

| | | |
|---|---|---|
| A right angle .................. | is | 90° |
| Two right angles .............. | " | 180° |
| Three right angles .............. | " | 270° |
| Half a right angle .............. | " | 45° |
| One third of a right angle ....... | " | 30° |
| Two thirds of a right angle ...... | " | 60° |

(13.) If two diameters of a circle be drawn at right angles with each other, as in the adjoining figure, it is obvious that the entire space will be divided into four equal portions, each being a right angle. Therefore the entire circumference may, with great propriety, be taken as the measure of 360°, or four right angles. Any fractional part of the circumference will be the measure of a like fractional part of 360°; thus, one fourth of the circumference is the measure of 90°, or one right angle.



The magnitude of the circumference has nothing to do with the magnitude of the angles, since their magnitudes depend wholly upon the fractional parts of the whole angular space about the centre C.

(14.) In the useful arts, all cutting tools have their edges formed into angles of various magnitudes, according to the materials to be cut. As a general rule, the softer the material to be divided, the more acute is the angle of the cutting edge. Chisels for cutting wood are formed with an angle of about 30°; those for cutting iron, at from 50° to 60°; and those for brass are 80° or more.

(15.) The angle which is by far the most extensively used in the arts, is the right angle. This is the angle of mechanical equilibrium, between the direction of any impact or pressure, and the resisting surface. A force cannot be wholly counteracted by a surface, unless the surface is exactly perpendicular to the direction of the force.

It is this principle which determines the erect position of natural structures of animals and plants; and it is by following out the architecture of nature, that artificial structures, raised by the hand of man, acquire stability and beauty. Buildings are erect, because the direction of their weight must be perpendicular to their support. A steeple or tower, which, by the yielding of the foundation, or any other cause, is out of the perpendicular, cannot be viewed without some sense of danger, and consequently some feelings of pain.

XII. Two straight lines are said to be *parallel*, when, being situated in the same plane, they cannot meet, how far soever, either way, both of them be produced. They are obviously every where equally distant.

(16.) The ordinary frames of windows and doors, and nearly all architectural frame work, consist of systems of parallel lines at right angles with each other. All fabrics produced in the loom, consist of two systems of parallel threads, crossing each other at right angles, so interlaced as to give strength and firmness to the cloth.

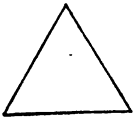
The railway consists of two or more parallel lines of iron bars, called *rails*.

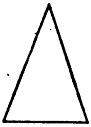
XIII. A *plane figure* is a limited portion of a plane. When it is limited by straight lines, the figure is called a *rectilineal figure*, or a *polygon;* and the limiting lines, taken together, form the *contour* or *perimeter* of the polygon.

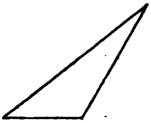
(17.) The surfaces of level fields, bounded by straight fences, are polygonal figures. Floors of buildings are polygons, usually having four sides.

XIV. The simplest kind of polygon is one having only three sides, and is called a *triangle.* A polygon of four sides is called a *quadrilateral;* that of five sides is called a *pentagon;* that of six sides is called a *heptagon;* and so on for figures of a greater number of sides.

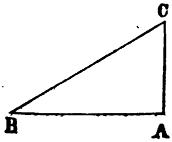
XV. A triangle having the three sides equal, is called an *equilateral triangle;* one having two sides equal, is called an *isosceles triangle;* and one having no two sides equal, is called a *scalene triangle.*



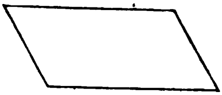




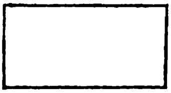
XVI. A triangle having a right angle, is called a *right-angled triangle.* The side opposite the right angle is called the *hypothenuse.* Thus BAC is a right-angled triangle, right-angled at A; the side BC is the hypothenuse.



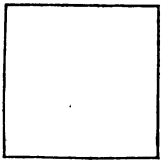
XVII. When the opposite sides of a quadrilateral are parallel, the figure is called a *parallelogram*.



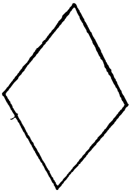
XVIII. When the four angles of a parallelogram are right angles, the figure is called a *rectangle*.



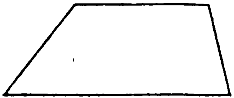
XIX. When the four sides of a rectangle are equal, the figure is called a *square*.



XX. When the four sides of a parallelogram are equal, and the angles not right, the figure is called a *rhombus*.



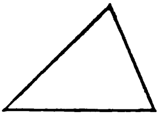
XXI. When only two sides of a quadrilateral are parallel, the figure is called a *trapezoid*.

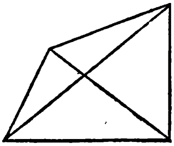


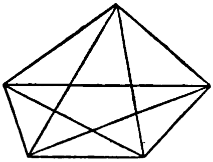
(18.) From the above definitions, it will be seen that the quadrilateral includes the parallelogram, the rectangle, the square, the rhombus, and the trapezoid; the parallelogram includes the rectangle, the square, and the rhombus; and the rectangle includes the square.

Nearly all architectural structures, such as doors, windows, floors, and the sides of houses, are of the rectangular form. Among the different triangles employed in architecture and carpentry, the isosceles is most frequently to be found. It is the form usually given to the roofs of buildings, and to the pediment which surmounts and adorns porticos, doors and windows.

XXII. A *diagonal* of a polygon is a line joining the vertices of two angles, not adjacent.



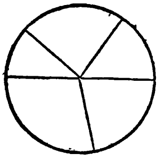




(19.) From the above definitions, in connection with the diagrams, it will be readily seen that the triangle has no diagonal, the quadrilateral has two diagonals, the pentagon has five, and so on for polygons of a greater number of sides.

The number of diagonals of a polygon of $n$ sides is given by this algebraic expression, $\frac{1}{2} n ( n - 3)$. [See *Elements of Algebra*, Art. 178.]

XXIII. A *circle* is a plane figure bounded by one line, which is called the *circumference;* and is such that all straight lines drawn from a certain point within the circle to the circumference, are equal to one another. This point is called the



*centre* of the circle. One of the equal lines drawn from the centre of a circle to its circumference, is called a *radius.* The line passing through the centre, and terminating each way in the circumference, is called a *diameter.*

## DEFINITION OF TERMS.

1. An *axiom* is a self-evident proposition.

2. A *theorem* is a truth, which becomes evident by means of a train of reasoning called a *demonstration.*

3. A *problem* is a question proposed, which requires a solution.

4. A *lemma* is a subsidiary truth, employed for the demonstration of a theorem, or the solution of a problem.

5. A *corollary* is an obvious consequence deduced from one or several propositions.

6. A *scholium* is a remark on one or several preceding propositions, which tends to point out their connection, their use, their restriction, or their extension.

7. A *postulate* is a problem, the method of solving which is obvious. It is therefore assumed or taken for granted by the geometer.

## AXIOMS.

I. Things which are equal to the same thing, are equal to each other.

II. When equals are added to equals, the wholes are equal.

III. When equals are taken from equals, the remainders are equal.

IV. When equals are added to unequals, the wholes are unequal.

V. When equals are taken from unequals, the remainders are unequal.

VI. Things which are double of the same or equal things, are equal.

VII. Things which are halves of the same thing, are equal.

VIII. Every whole is equal to all its parts taken together, and greater than any of them.

IX. Things which coincide, or fill the same space, are identical.

X. All right angles are equal to one another.

### POSTULATES.

I. To draw a straight line from any one point to any other point.

II. To produce a terminated straight line to any length.

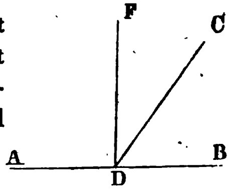
III. To describe the circumference of a circle, from any centre, with any radius, or, in other words, at any distance from that centre.

# PROPOSITIONS.

## PROPOSITION I.

**Theorem.** *When a straight line meets another straight line, the sum of the two adjacent angles, thus formed, is equal to two right angles.*

Let the straight line AB be met by the straight line CD at the point D. Then will the two adjacent angles ADC, BDC be together equal to two right angles.



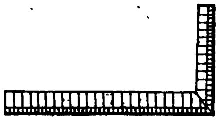
Suppose the line DF to be at right angles to AB (Def. X). The angle ADC is composed of the two angles ADF and FDC : therefore the sum of the two angles ADC, CDB is equal to the sum of the three angles ADF, FDC, CDB (Ax. II); of which the first ADF is a right angle (Def. X), and the sum of the other two FDC and CDB composes the right angle FDB : therefore the sum of the two angles ADC and BDC is equal to two right angles.

*Cor.* 1. Hence, also, conversely, if the two angles ADC, BDC, on the same side of the line AB, make up together two right angles, then AD and DB will form a continued straight line AB.

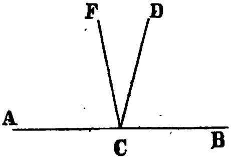
*Cor.* 2. Hence, all the angles which can be made at any point D, by any number of lines on the same side of AB, are together equal to two right angles.

*Cor.* 3. And as all the angles that can be made on the other side of AB are also equal to two right angles; therefore all the angles that can be made around the point D, by any number of lines, are equal to four right angles.

(20.) The instrument called a ***square,*** which is extensively used in the arts for tracing lines at right angles to each other, consists of two flat rulers placed at right angles as in the adjoining figure. When much precision is required, great care should be taken by those purchasing this instrument, to test its accuracy. The preceding Proposition suggests a very simple and sure means of making such test.



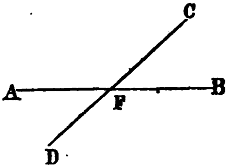
Suppose the straight line AB to be the edge of a board, or any other plane surface. Apply one side of the square so as to coincide with AC; then, along the other edge, upon the surface, draw the line CD. Now, reversing the square, apply the first side so as to coincide with BC; and then, along the second side, trace upon the surface the line CF. If the square is perfectly accurate, it is obvious that the lines CD, CF will coincide.



This method not only detects an error in the instrument, when it exists, but also shows the amount of error; that is, how much the angle of the square exceeds or falls short of being a right angle, or of containing just 90°.

## PROPOSITION II.

THEOREM. *When two lines intersect each other, the opposite angles are equal.*



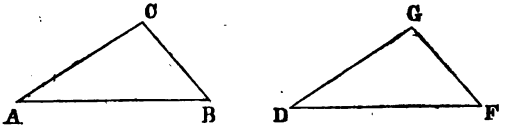
Let the two lines AB, CD intersect at the point F; then will the angle AFC be equal to BFD, and the angle AFD equal to BFC.

For, since the line CF meets the line AB, the two angles AFC, BFC, taken together, are equal to two right angles (Prop. I). In like manner, the line BF, meeting the line CD, makes the sum of the two angles BFC, BFD equal to two right angles. Therefore the sum of the two angles AFC, BFC is equal to the sum of the two BFC, BFD (Ax. I). And if the angle BFC, which is common, be taken away from each of these equals, the remaining angle AFC will be equal to the remaining angle BFD (Ax. III). And in the same manner it may be shown that the angle AFD is equal to BFC.

## PROPOSITION III.

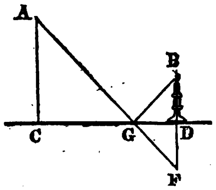
THEOREM. *If two triangles have two sides and the included angle of the one, equal to the two sides and the included angle of the other, the triangles will be identical, or equal in all respects.*

In the two triangles ABC, DFG, if the side CA be equal to the side GD, and the side CB equal to the side GF, and the angle C equal to the angle G; then will the two triangles be identical, or equal in all respects.

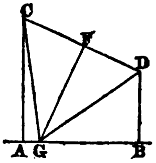


For, conceive the triangle ABC to be placed upon the triangle DFG, in such a manner that the point C may coincide with the point G, and the side CA with the equal side GD. Then, since the angle G is equal to the angle C, the side CB will take the direction of the side GF. Also CB being equal to GF, the point B will coincide with the point F; consequently the side AB will coincide with DF. Therefore the two triangles are identical, and have all their other corresponding parts equal (Ax. IX), namely, the side AB equal to the side DF, the angle A equal to the angle D, and the angle B equal to the angle F.

(21.) Let BD be a candlestick with a candle, whose flame is at the point B, standing perpendicularly on a plane mirror CD. Let A be the position of the eye at the height of CA, above the mirror. It is required to find the point G at which the light is reflected so as to enter the eye, the angle AGC being equal to the angle BGD.



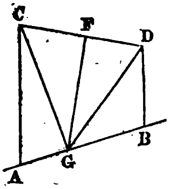
*Solution.* Produce BD until DF is equal to BD: draw AF, cutting the mirror at G, and G will be the point required. For, comparing the triangle GFD with GBD, we have DF equal to DB, DG common, and the angle GDF equal to GDB, each being a right angle; therefore those triangles are equal (Prop. III) : consequently the angle DGB is equal to the angle DGF; but DGF is equal to the opposite angle AGC (Prop. II); hence the angle BGD is equal to AGC.

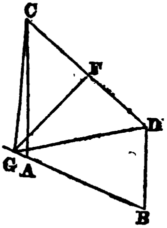


(22.) Suppose AC and BD to represent two trees standing on the horizontal plane AB, it is required to find a point in this plane equally distant from the tops C & D.

*Solution.* Join CD, and bisect it by the perpendicular FG; then will the point G be the point sought. For, if we join GC, GD, we shall have the triangle GFC equal to GFD, since the side FC is equal to FD, the side FG common, and the angle CFG equal to DFG, each being a right angle. Therefore (Prop. III), the triangle GFC is equal to GFD; consequently GC is equal to GD.

(23.) It is also obvious that, in this question, the above method of construction will apply when the two trees do not stand upon a horizontal plane.



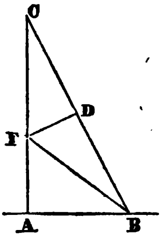


The above construction will also apply in case the ground is horizontal, and the trees, instead of being vertical, are oblique.

(24.) The foregoing are only particular cases of the following more general proposition:

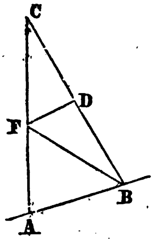
**Problem.** *To find a point in a given straight line, equally distant from two given points.*

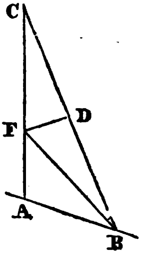
(25.) Suppose AC to be a tree, standing on the horizontal plane AB; it is required to find at what point it must be broken, so that, by falling, the top may strike the ground at B.



*Solution.* Join CB, and bisect it by the perpendicular DF; then will F be the point at which the tree must break. For, joining BF, and comparing the triangle FBD with the triangle FCD, we see that the side DB is equal to DC, the side FD common, and the contained angle FDB equal to FDC, each being a right angle. Therefore (Prop. III) the triangle FBD is equal to FCD; consequently FB is equal to FC.

The solution is effected in the same way when the tree is supposed to stand upon an inclined surface, as upon the side of a hill.

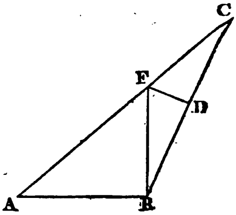




(26.) The general proposition, of which the foregoing are particular cases, may be thus given:

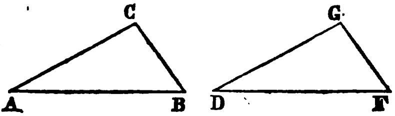
**Problem.** *Given the base of a triangle, one of the angles at the base, and the sum of the other two sides, to construct the triangle.*

*Construction.* Make AB equal to the given base; draw AC, making the angle BAC equal to the given angle, and AC equal to the sum of the other two sides; join BC, and bisect it by the perpendicular DF; finally join BF, and ABF will be the triangle required. This is obvious from what has already been done.



### PROPOSITION IV.

THEOREM. *If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, the triangles will be identical, or equal in all respects.*



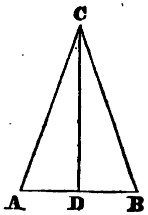
In the two triangles ABC, DFG, if the angle A is equal to the angle D, the angle B equal to the angle F, and the side AB equal to the side DF; then will the triangles be identical, or equal in all respects.

For, conceive the triangle ABC to be placed on the triangle DFG, in such a manner that the side AB may coincide with the equal side DF. Then, since the angle D is equal to the angle A, the side AC will take the direction of the side DG; also, since the angle F is equal to the angle B, the side BC will take the direction of the side FG; consequently the point C must coincide with the point G. Therefore the two triangles are identical (Ax. IX), having the two sides AC and BC respectively equal to DG and FG, and the remaining angle C equal to the remaining angle G.

### PROPOSITION V.

THEOREM. *In an isosceles triangle, the angles at the base are equal; or, if a triangle have two sides equal, the angles opposite those sides will be equal.*

If the triangle ABC have the side AC equal to the side BC, then will the angle B be equal to the angle A.



For, conceive the angle C to be bisected, or divided into two equal parts, by the line CD, making the angle ACD equal to the angle BCD. Then the two triangles ADC and BDC have two sides and the included angle of the one equal to the two sides and the included angle of the other, namely, the side AC equal to BC, the angle ACD equal to the angle BCD, and the side CD common; therefore the two triangles are identical, or equal in all respects (Prop. III), and consequently the angle B is equal to the angle A.

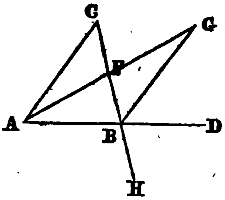
*Cor.* 1. Hence the line which bisects the vertical angle of an isosceles triangle, bisects the base, and is also perpendicular to it.

*Cor.* 2. It also appears that every equilateral triangle is equiangular, or has all its angles equal.

### PROPOSITION VI.

THEOREM. *When one side of a triangle is produced, the exterior angle is greater than either of the two interior and opposite angles.*

Let ABC be a triangle, having the side AB produced to D; then will the exterior angle CBD be greater than either of the interior and opposite angles BAC or BCA.

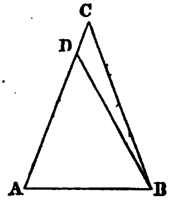


For, conceive the side BC to be bisected in the point F, and draw the line AF and produce it until FG is equal to AF; and join BG. Now, in the two triangles AFC and GFB, the sides FA and FC are respectively equal to the sides FG and FB, and the opposite angles AFC and GFB are equal (Prop. II); therefore these two triangles are equal in all respects (Prop. III), and we have the angle ACF equal to the angle GBF; consequently the exterior angle CBD, being greater than GBF, is greater than the interior angle BCA.

In like manner, if CB be produced to H, and AB be bisected, it may be shown that the exterior angle ABH, or its equal CBD, is greater than the other interior angle BAC.

### PROPOSITION VII.

THEOREM. *When a triangle has two of its angles equal, the sides opposite to them are also equal, and the triangle will be isosceles.*

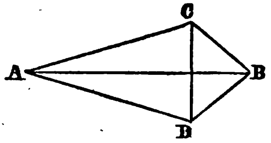


In the triangle ABC, let the angle CAB be equal to the angle CBA; then will the side CB be equal to CA.

For, if these sides are not equal, one must be greater than the other: let CA be greater than CB; then take AD equal to BC, and join DB. Now in the two triangles DAB and CBA, we have DA equal to CB by construction, the side AB common, and the angle CAB equal to the angle CBA by hypothesis; therefore two sides and the included angle of the one are respectively equal to two sides and the included angle of the other; consequently the triangle DAB is equal to the triangle CBA (Prop. III). But a part cannot be equal to the whole (Ax. VIII); hence there can be no inequality between the sides CB and CA, and therefore the triangle is isosceles.

### PROPOSITION VIII.

THEOREM. *When two triangles have all the three sides of the one equal to all the three sides of the other, the triangles will be identical, or equal in all respects.*



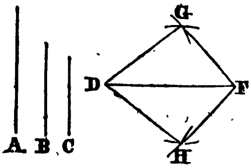
Let the two triangles ABC, ABD have their sides respectively equal, namely, AB equal to AB, AC equal to AD, and BC equal to BD; then will these triangles be identical.

For, conceive the two triangles to be joined together by their longest equal sides, and draw the line CD; then, in the triangle ACD, since AC is equal to AD, we have the angle ACD equal to the angle ADC (Prop. v). In like manner, in the triangle BCD, since BC is equal to BD, we have the angle BCD equal to the angle BDC. Hence the angle ACB, which is the sum of ACD and BCD, is equal to the angle ADB, which is the sum of ADC and BDC. Since, then, in the triangle ACB, we have the two sides AC and BC, and their included angle ACB, equal respectively to the two sides AD and BD, and their included angle ADB, of the triangle ADB, it therefore follows that these triangles are identical (Prop. III).

### PROPOSITION IX.

**Problem.** *Three straight lines* A, B, C, *each of which is less than the sum of the two others, being given, to construct a triangle whose sides shall be respectively equal to them.*

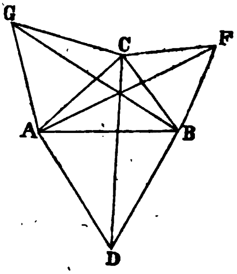
Make the line DF equal to the line A; and with D and F as centres, and with radii equal respectively to the lines B and C, describe arcs intersecting at the point G (Post. III). Join DG, FG (Post. I), and the triangle DGF will be the triangle required, since the three sides are equal to the three lines A, B, C.



*Scholium.* It is obvious that the arcs described from D and F as centres, will intersect in two points G and H, thus giving two triangles DGF and DHF; but these two triangles, having the three sides of the one equal to the three sides of the other, are identical (Prop. VIII).

If two of the given lines are equal, the triangle will be *isosceles;* when all the lines are equal, it will be *equilateral.*

(27.) Theorem. *If, on the three sides of any triangle, equilateral triangles be constructed externally to the given triangle, then will the straight lines drawn from the vertices of the equilateral triangles to the opposite angles of the given triangle be equal.*



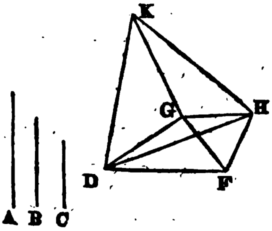
Let ABC be the given triangle, upon whose sides the equilateral triangles ADB, BFC, CGA, are described; then will the lines DC, FA, GB, be equal.

Comparing the two triangles ABF, DBC, we have AB = DB; BF = BC, being sides of equilateral triangles. The angles CBF, ABD are equal, being angles of equilateral triangles; to each of these angles add the angle ABC, and we have the angle ABF = DBC. Therefore the two triangles ABF and DBC have the two sides and the included angle of the one equal to the two sides and the included angle of the other; they are therefore identical (Prop. III), and consequently DC is equal to FA.

By comparing the triangles ACF and GCB, it may be shown that they also are equal, and consequently FA is equal to GB.

It might, moreover, be shown that these lines all intersect in the same point.

(28.) PROBLEM. *Given the lengths of three lines drawn from a point within an equilateral triangle, to the three corners; to find the side of the triangle.*

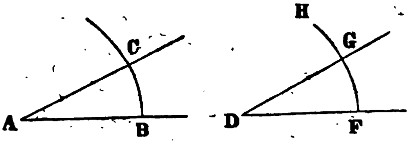


Let A, B, C, be the three given lines. With these lines construct the triangle DFG (Prop. IX); upon either of the sides, as FG, construct the equilateral triangle FGH; join DH, and it will be a side of the equilateral triangle sought.

For, on DH construct the equilateral triangle DHK, and join GK; then comparing the two triangles DFH & KGH, we see that DH and FH of the one are respectively equal to KH and GH of the other, being sides of equilateral triangles; the angle FHG is equal to DHK, each being an angle of an equilateral triangle (Prop. V, Cor. 2); from each take the angle DHG, and we have FHD equal to GHK; therefore the two triangles DFH and KGH are equal (Prop. III), and consequently GK is equal to DF. Hence the three lines GK, GD, GH, are respectively equal to the three DF, DG, GF, which are equal to the given lines A, B, C; which proves DH to be the side of the equilateral triangle sought.

## PROPOSITION X.

**Problem.** *At a given point* D, *in a given line* DF, *to make an angle equal to a given angle* BAC.

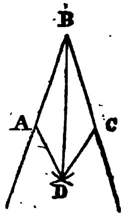


With any radius describe arcs from A & D as centres (Post. III); the first BC, meeting AB, AC, at B and C; and the second FH meeting DF at F. With F as a centre, and a radius equal to the distance from B to C, describe an arc (Post. III) to meet FH at G. The line DG being drawn, will make the angle FDG equal to BAC.

This is an application of Prop. IX, and the equality of the angles will follow from Prop. VIII. For, drawing lines from B to C, and from F to G (Post. I), we have the three sides of the triangle ABC equal to the three sides of the triangle DFG; therefore (Prop. VIII) the angle FDG will be equal to the angle BAC.

### PROPOSITION XI.

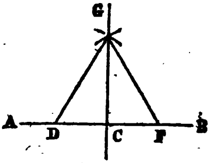
**Problem.** *To bisect a given angle* ABC.



Take any equal distances BA, BC, on the sides containing the angle; and with A and C as centres, and any equal radii, describe arcs intersecting each other at D; then, BD being drawn, it will bisect the angle ABC. For, drawing AD, CD, the three sides of the triangle ABD are equal respectively to the three sides of the triangle CBD; hence (Prop. VIII) the angles ABD and CBD are equal.

### PROPOSITION XII.

**Problem.** *Through a given point* C, *in a given line* AB, *to draw a perpendicular.*

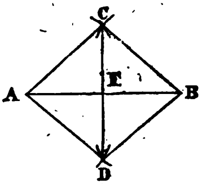


Take equal distances CD, CF, on each side of the given point; and with any equal radii, describe arcs (Post. III) meeting at G. Join GC (Post. I), and it will be the perpendicular required.

For, conceive DG, FG, to be drawn; then the equality of the sides of the triangles DGC, FGC, give the angles at C equal (Prop. VIII), and hence GC is perpendicular to AB (Def. X).

### PROPOSITION XIII.

**Problem.** *To bisect a given straight line* AB.

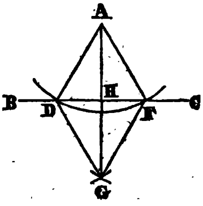


With A and B as centres, with any convenient equal radii, describe arcs (Post. III) intersecting at C and D. Draw CD (Post. I), and it will be perpendicular to AB, and will bisect it at the point F.

For, by joining AC and BC, AD and BD, we shall have two triangles CAD and CBD, with all the sides of the one equal respectively to all the sides of the other; consequently the angle ACF is equal to BCF (Prop. VIII). Hence since the line CF bisects the vertical angle of the isosceles triangle ACB, it bisects the base AB (Prop. V, Cor. 1).

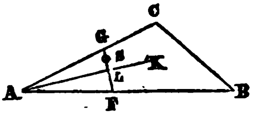
### PROPOSITION XIV.

**Problem.** *From a given point* A *without a given line* BC, *to draw a line perpendicular to* BC.



With A as a centre, with any convenient radius, describe an arc (Post. III) cutting BC in the two points D and F; and with D and F as centres, and with equal radii, describe arcs (Post. III) intersecting at G. Then AG being drawn, cutting BC at H, will be the perpendicular required.

For, joining DA and FA, DG and FG, we have all the sides of the triangle ADG equal respectively to all the sides of the triangle AFG; consequently the angle DAH is equal to to FAH (Prop. viii); and since AH bisects the vertical angle of the isosceles triangle DAF, it is perpendicular to the base DF (Prop. v, Cor. 1).



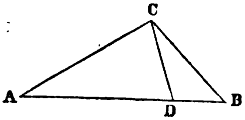
(29.) Let ABC be a triangular field, and S a well within it; it is required to divide the field into two parts by a line passing through S, so that the distances AF, AG shall be equal.

*Solution.* Draw the line AK, bisecting the angle BAC (Prop. xi); and through the point S draw GF perpendicular to AK (Prop. xiv), and it will be the line required.

For, comparing the two triangles ALF, ALG, we have the angle LAF equal to the angle LAG by construction; the angle ALF equal to the angle ALG, each being a right angle; and the side AL common: hence we have two angles and the interjacent side of the one triangle equal to two angles and the interjacent side of the other; consequently the triangles are equal, and AF = AG (Prop. iv).

### PROPOSITION XV.

Theorem. *The greater side of every triangle is opposite the greater angle; and the greater angle is opposite the greater side.*



In the triangle ABC, let the side AB be greater than the side AC; then will the angle ACB opposite the greater side AB be greater than the angle ABC opposite the less side.

For, from the greater side AB take a portion AD equal to the less side AC, and join CD. Now, since the angle ADC is an exterior angle in reference to the triangle CDB, it is greater than the interior angle ABC (Prop. VI); but since the side AD is equal to the side AC, the angle ACD is equal to ADC (Prop. V), and consequently the angle ACD is greater than ABC; and since the angle ACD is only a part of the angle ACB, much more then must the whole angle ACB be greater than ABC.

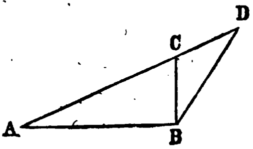
Again, conversely, if the angle ACB is greater than the angle ABC, then will the side AB, opposite the former, be greater than the side AC opposite the latter.

For, if AB is not greater than AC, it must be either equal to it, or less than it. But it cannot be equal; for then the angle ACB would be equal to the angle ABC (Prop. V), which, by supposition, is not the case. Neither can it be less; for then the angle ACB would be less than the angle ABC, by the first part of this proposition; which is also contrary to the supposition. Hence, since the side AB is neither equal to AC, nor less than it, it must of necessity be greater.

### PROPOSITION XVI.

THEOREM. *The sum of any two sides of a triangle is greater than the third side.*

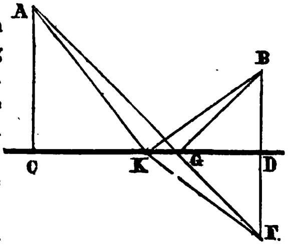
In the triangle ABC, the sum of the two sides AC and BC is greater than the third side AB.



For, produce AC till CD be equal to CB, or AD equal to the sum of the two AC, CB : join BD. Then because CB is equal to CD, the angle D is equal to CBD (Prop. v). But the angle ABD is greater than the angle CBD; consequently it must be greater than D. And since the greater side of any triangle is opposite the greater angle (Prop. xv), the side AD of the triangle ABD is greater than the side AB. But AD is equal to the sum of AC, CD, or to the sum of AC, CB; therefore AC + CB is greater than AB.

*Cor.* The shortest distance between two points is a straight line drawn from one point to the other.

(30.) This is the celebrated proposition which the Epicureans derided, as being manifest even to asses. They supposed an ass would know that it was further around the corner of a field, than across the same corner.



(31.) Let DB be a candlestick with a candle, whose flame is at B, standing perpendicularly on a plane mirror CD. Let A be the position of the eye at the height CA above the mirror. Required to prove that the light, passing from the candle to the mirror, and thence to the eye, obeying the law of nature, that is, making the angle of reflection equal to the angle of incidence, takes the minimum route. In other words, prove that the sum of the two lines BG, GA is less than the sum of any other two lines drawn from B and A to meet in a point of the mirror, such as the two lines BK, KA.

Produce BD until DF is equal to BD, and join KF, and then as in (21) we can show that KF is equal to KB, so that AK + KB is equal to AK + KF; also AG + GB is equal to AG + GF, or to AF.

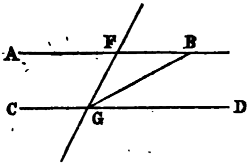
Now, by the above proposition, AF is less than AK + KF; therefore AG + GB is less than AK + KB.

The time required for the light to pass from B to A, by the reflection of the mirror CD, is, by the law of nature, the shortest possible.

### PROPOSITION XVII.

**Theorem.** ***When a line intersects two parallel lines, it makes the alternate angles equal to each other.***

Let the line FG cut the two parallel lines AB, CD; then will the angle AFG be equal to the alternate angle DGF.

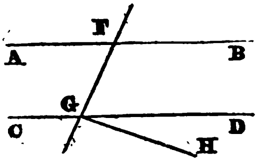


For, if they are not equal, one of them must be greater than the other: let it be DGF, for instance, which is the greater, if possible; and conceive the line GB to be drawn, cutting off the part or angle FGB equal to the angle AFG, and meeting the line AB in the point B. The angle AFG, being exterior in reference to the triangle GFB, is greater than the interior and opposite angle FGB (Prop. VI). Hence the angle AFG is greater and less than FGB, at the same time, which is impossible. Therefore the angle FGD is not unequal to the alternate angle AFG; that is, they are equal to each other.

*Cor.* Straight lines which are perpendicular to one of of two parallel lines, are also perpendicular to the other.

## PROPOSITION XVIII.

THEOREM. *When a line, cutting two other lines, makes the alternate angles equal to each other, those two lines are parallel.*

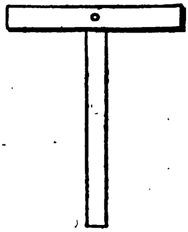


Let the line FG, cutting the two lines AB, CD, make the alternate angles AFG, DGF, equal to each other; then will AB be parallel to CD.

For if they are not parallel, let some other line, as GH, be parallel to AB. Then, because of these parallels, the angle AFG is equal to the alternate angle HGF (Prop. XVII); but the angle AFG is equal to the angle DGF; therefore the angle DGF is equal to the angle HGF (Ax. I); that is, a part is equal to the whole, which is impossible. Therefore no line drawn through the point G, except the line CD, can be parallel to AB.

*Cor.* Those lines which are perpendicular to the same line, are parallel to each other.

(32.) The principle contained in the Corollary, that lines which are perpendicular to the same line are parallel to each other, has a practical application in the use of the instrument called the T-*square.* It consists of two straight rulers fixed at right angles to each other, forming a sort of double square. A straight line being drawn in a direction perpendicular to that in which the parallels are required to be drawn, the cross-piece of the T-square is laid upon this line, and the piece at right angles to it gives the direction of the parallels. The ruler being moved along the paper, keeping the cross-piece coincident

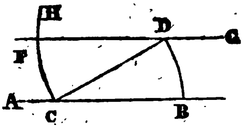


**with the line first drawn, any number of parallel lines may thus be formed.**

**By means of a movable joint, the rulers which compose this kind of square can be made to form any angle whatever with each other, which, in practice, is found to be quite convenient. To suit this more general case, the preceding corollary might read "Those lines which make the same angle with the same line, are parallel."**

### PROPOSITION XIX.

PROBLEM. *Through a given point* D, *to draw a line parallel to a given line* AB.

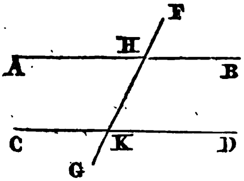


Draw any line DC, meeting AB at C; and with C and D as centres, and CD as a radius, describe the arcs (Post. III) DB, CH, the former meeting AB at B. Then with C as a centre, and with a radius equal to the distance from B to D, describe an arc to meet the arc CH at the point F. The line DF, being drawn, will be parallel to AB.

For (Prop. VIII), the alternate angles FDC, DCB are equal; and therefore (Prop. XVIII) the line FD is parallel to AB.

### PROPOSITION XX.

THEOREM. *When a straight line cuts two parallel lines, the exterior angle is equal to the interior and opposite one, on the same side; and the two interior angles, on the same side, are together equal to two right angles.*



Let the line FG cut the two parallel lines AB, CD; then will the exterior angle FHB be equal to the interior and opposite angle HKD, on the same side of the line FG; and the two interior angles BHK, DKH, taken together, will be equal to two right angles.

For, since the two lines AB, CD are parallel, the angle AHK is equal to the alternate angle DKH (Prop. XVII); but the angle AHK is equal to the opposite angle FHB (Prop. II); therefore the angle FHB is equal to the angle HKD (Ax. I).

Again, because the two adjacent angles BHF, BHK are together equal to two right angles (Prop. I), of which the angle FHB has been shown to equal the angle HKD; therefore the two angles BHK, DKH, taken together, are also equal to two right angles.

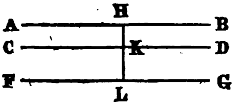
*Cor.* 1. And, conversely, if one line meet two other lines, making the exterior angle equal to the interior and opposite one, those two lines will be parallels.

*Cor.* 2. If a line, cutting two other lines, make the sum of the two interior angles, on the same side, less than two right angles, those two lines will not be parallel, and consequently will meet each other when produced.

## PROPOSITION XXI.

**Theorem.** *Those lines which are parallel to the same line, are parallel to each other.*

Let the lines AB, CD be each parallel to FG; then will the lines AB, CD be parallel to each other.

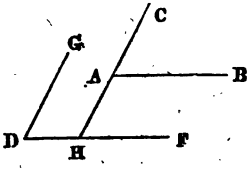


For, let the line HL be perpendicular to FG; then will this line be also perpendicular to both the lines AB, CD (Prop. xvii, Cor.), and consequently the two lines AB, CD are parallels (Prop. xviii, Cor.).

## PROPOSITION XXII.

**Theorem.** *If two angles have their sides parallel, and lying in the same direction, the two angles will be equal.*

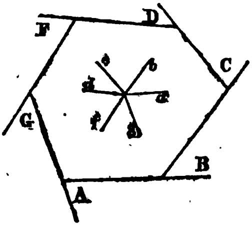
Let BAC and FDG be the two angles, having AB parallel to DF, and AC parallel to DG; then will the angles be equal.



For, produce CA, if necessary, till it meets DF in H. Then, since AB is parallel to DF, the angle BAC is equal to FHA (Prop. xx); and since HC is parallel to DG, the angle FHA is equal to FDG (Prop. xx); hence the angle BAC is equal to the angle FDG (Ax. I).

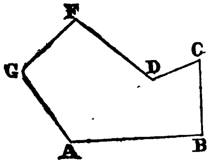
## PROPOSITION XXIII.

THEOREM. *When each side of a polygonal figure is produced, in the same direction, the sum of all the exterior angles will be equal to four right angles.*



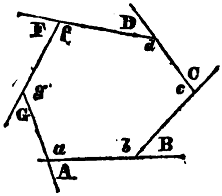
For, if from any point in the same plane, straight lines be drawn parallel to the sides of the figure, the angles contained by the straight lines about that point will be equal to the exterior angles of the figure (Prop. XXII), each to each, because their sides are parallel to the sides of the figure. Thus the angles *a*, *b*, *c*, etc. are respectively equal to the exterior angles A, B, C, etc.; but the former angles are together equal to four right angles (Prop. I, Cor. 3); therefore all the exterior angles of the figure are together equal to four right angles.

*Scholium.* This proposition must be restricted to the case in which the polygon is strictly convex. A convex polygon may be defined to be one such that no side, by being produced in either direction, can divide the polygon. The polygon ABCDFG is not convex, since it may be divided by producing either of the sides CD or FD. This polygon is said to have a re-entering angle at D.



## PROPOSITION XXIV.

**Theorem.** *In any convex polygon, the sum of all the interior angles, taken together, is equal to twice as many right angles as the polygon has sides, wanting four right angles.*

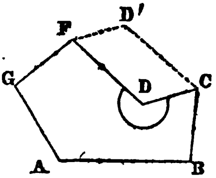


Let ABCDFG be a convex polygon. Conceive the sides to be produced all in the same direction, forming exterior angles, which we will denote by the capital letters A, B, C, etc., while their corresponding interior angles are denoted by the small letters *a*, *b*, *c*, etc. Now any exterior angle, together with its adjacent interior angle, as A + *a*, is equal to two right angles (Prop. i); therefore the sum of all the interior angles, together with all the exterior angles, is equal to twice as many right angles as the polygon has sides; but the sum of all the exterior angles is equal to four right angles (Prop. xxiii); therefore the sum of all the interior angles is equal to twice as many right angles as the polygon has sides, wanting four right angles.

*Cor.* 1. In any triangle, the sum of all the three angles is equal to two right angles.

*Cor.* 2. In any quadrilateral, the sum of all the four interior angles is equal to four right angles.

(33.) In order that this proposition may hold good in polygons having re-entering angles, as in the case of the polygon ABCDFG (next page), which has a re-entering angle at D, we must take, for the interior angle at D, the angle which remains after subtracting the angle CDF from four right angles; that is, we must consider this angle as exceeding two right angles.

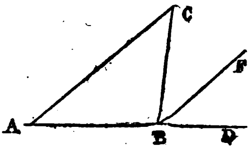


For, drawing CD' and FD' respectively parallel to DF and DC, we shall form a parallelogram CDFD'; and the figure ABCD'FG will be a convex polygon, having the same number of sides as the original polygon. And it is moreover obvious that the sum of all the interior angles of this convex polygon is the same as the sum of all the angles of the original polygon, if we consider the angle at D as what remains after taking the angle CDF from four right angles.

(34.) A practical application of the principle contained in this proposition is made by the land surveyor, as a test of the accuracy of his work. By means of the courses of the different sides of his field, he can determine the magnitude of all the interior angles, estimated in degrees and minutes; he takes the sum of all these angles, and observes whether it is such as to correspond with the above conditions. If the number of sides of the field be denoted by $n$, then the sum of all the interior angles ought to be equal to $(2n-4)$ times a right angle; thus, if there are 7 sides in the field, all the angles ought to amount to 10 right angles, or 900°. If the field contain re-entering angles (33), it will be necessary to modify somewhat this method.

### PROPOSITION XXV.

**Theorem.** *When one side of a triangle is produced, the exterior angle is equal to both the interior and opposite angles taken together.*



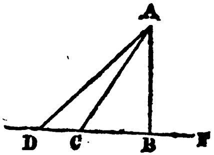
Let the side AB of the triangle ABC be produced to D; then will the exterior angle CBD be equal to the sum of the two interior and opposite angles A and C.

For, conceive BF to be drawn parallel to AC; then BC, meeting the two parallel lines AC, BF, makes the alternate angles C and CBF equal (Prop. XVII); and AD, cutting the same two parallels AC, BF, makes the interior and exterior angles on the same side, A and FBD, equal to each other (Prop. XX); therefore, by equal additions, the sum of the two angles A and C is equal to the sum of the angles CBF and FBD, that is, to the whole angle CBD (Ax. II).

### PROPOSITION XXVI.

**Theorem.** *A perpendicular is the shortest line that can be drawn from a given point to a line of indefinite length; and of any other lines drawn from the same point, those that are nearest the perpendicular are shorter than those more remote.*

If AB, AC, AD, be lines drawn from the given point A, to the indefinite line DF, of which AB is perpendicular; then will the perpendicular AB be less than AC, and AC less than AD.



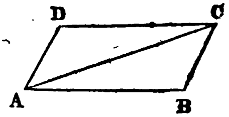
For, the angle B being right, the angle C must be acute (Prop. VI), and therefore less than the angle B; but the lesser side of a triangle is situated opposite the lesser angle (Prop. XV), therefore the side AB is less than the side AC.

Again, the angle ACB being acute, as before, the adjacent angle ACD will be obtuse (Prop. I); consequently the angle ADC is acute (Prop. VI), and therefore it is less than the angle ACD; and since the lesser side is opposite the lesser angle, the side AC must be less than the side AD.

*Cor.* A perpendicular measures the shortest distance of a point from a line.

### PROPOSITION XXVII.

THEOREM. *The opposite angles and sides of a parallelogram are equal to each other; and the diagonal divides it into two equal triangles.*



Let ABCD be a parallelogram, of which the diagonal is AC; then will its opposite sides and angles be equal to each other, and the diagonal will divide it into two equal triangles.

For, since the sides AB and DC are parallel, as also the sides AD and BC (Def. XVII), and the line AC meets them, therefore the alternate angles are equal (Prop. XVII), namely, the angle BAC to the angle DCA, and the angle BCA to the angle DAC; hence the two triangles, having two angles of the one equal to two angles of the other, have also their third angles equal (Prop. XXIV, Cor. 1), namely, the angle B equal to the angle D, which are two of the opposite angles of the parallelogram. Also if to the equal angles BAC, DCA be added the equal angles DAC, BCA, the wholes will be equal (Ax. II), namely, the whole angle BAD to the whole angle DCB, which are the other two opposite angles of the parallelogram.

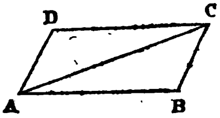
Again, since the side AC is common to the two triangles ACB and CAD, we have two angles and the interjacent side of the one equal to two angles and the interjacent side of the other, and therefore the triangles are equal (Prop. IV). Therefore the side AB is equal to its opposite side CD, and BC equal to its opposite side DA, and the whole triangle ABC equal to the whole triangle BCD.

*Cor.* 1. If one angle of a parallelogram be a right angle, all the other three angles will also be right angles, and the parallelogram will be a rectangle (Def. XVIII).

*Cor.* 2. Hence, also, the sum of any two adjacent angles of a parallelogram is equal to two right angles.

### PROPOSITION XXVIII.

THEOREM. *Every quadrilateral, whose opposite sides are equal, is a parallelogram, or has its opposite sides parallel.*



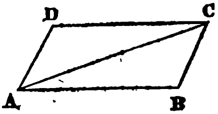
Let ABCD be a quadrilateral, having the opposite sides equal, namely, the side AB equal to DC, and AD equal to BC; then will these equal sides be also parallel, and the figure will be a parallelogram.

For, let the diagonal AC be drawn; then the triangles ABC, CDA, being mutually equilateral, are also mutually equiangular (Prop. VIII), or have their corresponding angles equal; consequently the opposite sides are parallel (Prop. XVIII), namely, the side AB parallel to DC, and AD parallel to BC, and the figure is a parallelogram (Def. XVII).

### PROPOSITION XXIX.

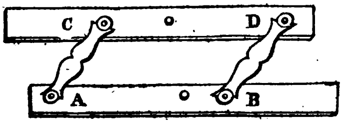
**Theorem.** *Those lines which join the corresponding extremes of two equal and parallel lines, are themselves equal and parallel.*

Let AB, DC be two equal and parallel lines; then will the lines AD, BC, which join their corresponding extremes, be also equal and parallel.



For, draw the diagonal AC; then, because AB and DC are parallel, the angle BAC is equal to its alternate angle DCA (Prop. XVII); hence the two triangles, having two sides and the included angle of the one equal to two sides and the included angle of the other, namely, the side AB equal to the side CD, the side AC common, and the contained angle BAC equal to the contained angle DCA, have also the remaining sides and angles respectively equal (Prop. III); consequently AD is equal to BC, and also parallel to it (Prop. XVIII).

(35.) The principle contained in this proposition has led to the construction of the *parallel ruler*. From the construction, which is readily understood by the aid of the adjoining figure, it is obvious that AB must in every position be parallel to DC, since AB and DC join the corresponding extremes of the two equal and parallel lines.



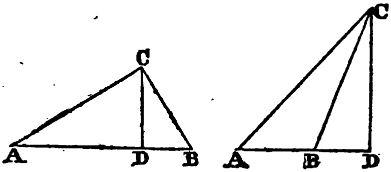
# BOOK SECOND.

## DEFINITIONS.

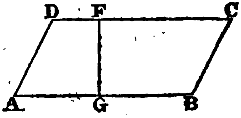
1. In a right-angled triangle, the side opposite the right angle is called the *hypothenuse;* and the other two sides are called the *legs*, and sometimes the *base* and *perpendicular*.

2. The *base* of any rectilineal figure, is the side on which the figure is supposed to stand.

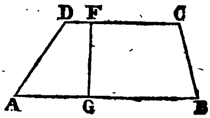
3. The *altitude* of a triangle is the perpendicular drawn from the vertex to the opposite side, or opposite side produced, considered as the base; thus, CD is the altitude of the triangle ABC.



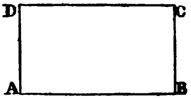
4. The altitude of a parallelogram is the perpendicular between two opposite sides considered as bases; thus, FG is the altitude of the parallelogram ABCD.



5. The altitude of a trapezoid is the perpendicular drawn between its two parallel sides; thus, FG is the altitude of the trapezoid ABCD.



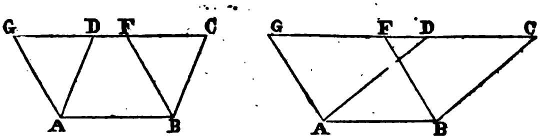
6. A rectangle is said to be *contained* by its adjacent sides; thus, the rectangle ABCD is contained by the sides AB, CD. For brevity, it is often referred to as the rectangle BA.AD.



### PROPOSITION I.

**THEOREM.** *Two parallelograms having the same base and same altitude, are equivalent.*

Let the two parallelograms ABCD, ABFG have the same base AB, and the same altitude; then will they be equivalent.



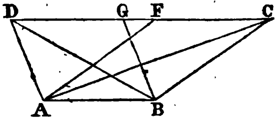
Since they have the same altitude, their upper bases will be in the same line GC, parallel to the common base AB. Now, by the nature of parallelograms, we have BC equal to AD, and BF equal to AG; we also have DC equal GF, each being equal to AB. If each of these equal lines be taken from the whole line GC, there will remain the line GD in the one case, equal to the line FC in the other (Ax. III); therefore the three sides of the triangle ADG are equal to the three sides of the triangle BCF, and consequently they are equal (B. I, Prop. VIII). If each of these equal triangles be taken from the whole space ABCG, there will remain the parallelogram ABCD in the one case, equal to the parallelogram ABFG in the other (Ax. III).

*Cor.* 1. Parallelograms having the same base, and being situated between the same parallels, are equivalent; for their altitudes will be perpendiculars between the two parallels, which are all equal by the definition of parallels.

*Cor.* 2. Parallelograms having equal bases and altitudes are equivalent; for they may be so applied as to have their equal bases coincide, and then by this proposition they will be equivalent.

### PROPOSITION II.

THEOREM. *Two triangles having the same base and same altitude, are equivalent.*



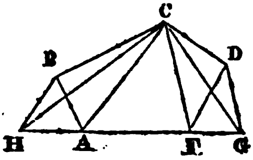
Let the two triangles ABC, ABD have the same base AB, and the same altitude; then will they be equivalent.

Since they have the same altitude, the line CD which joins their vertices will be parallel to the common base AB. Draw AF parallel to BC, and BG parallel to AD; thus forming the two parallelograms ABCF and ABGD, which are equivalent (B. II, Prop. I). The triangle ABC is one half of the parallelogram ABCF, and the triangle ABD is one half of the parallelogram ABGD (B. I, Prop. XXVII); therefore the triangle ABC is equivalent to the triangle ABD.

*Cor.* 1. Triangles having the same base, and situated between the same parallels, are equivalent; for the altitude is the perpendicular between the two parallels, which is every where equal.

*Cor.* 2. Triangles having equal bases and the same altitude, or being situated between the same parallels, are equivalent.

(36.) Problem. *To find a triangle that shall be equivalent to a given polygon.*



Let ABCDF be the given polygon. Draw the diagonal CF, cutting off the triangle CDF; through the point D, draw DG parallel to CF, and meeting AF produced; draw CG, and the polygon ABCDF will be equivalent to ABCG, which has one side less than the original polygon.

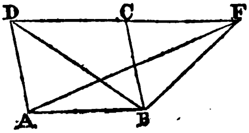
For, the triangles CDF, CGF have the common base CF; and they have the same altitude, since their vertices D and G are situated in the line DG which is parallel to the base CF; therefore these triangles are equivalent (B. II, Prop. II). To each add the figure ABCF, and we shall have the polygon ABCDF equivalent to the polygon ABCG.

For a similar reason, the triangle CAH may be substituted for the equivalent triangle CAB, and thus the pentagon ABCDF will be changed into an equivalent triangle HCG.

It is obvious the same process may be applied to a polygon having any number of sides; each step lessening the number of sides by one, until we finally arrive at an equivalent triangle.

## PROPOSITION III.

Theorem. *If a parallelogram and a triangle have the same base and the same altitude, the triangle will be half the parallelogram.*



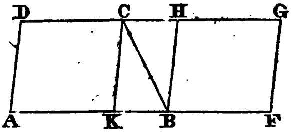
Let the parallelogram ABCD, and the triangle ABF, have the same base AB, and the same altitude; then will the triangle be half the parallelogram.

For, draw the diagonal BD of the parallelogram, dividing it into two equal parts (B. I, Prop. XXVII). Now, since the triangles ABF, ABD have the same base and the same altitude, they are equivalent (B. II, Prop. II); and since ABD is half the parallelogram ABCD, it follows that ABF is also half the same parallelogram.

*Cor.* A triangle is half the parallelogram having the same base, and being situated between the same parallels; for, being between the same parallels, they must have the same altitude, and by this proposition the triangle must be half the parallelogram.

### PROPOSITION IV.

THEOREM. *A trapezoid, or trapezium having two sides parallel, is equivalent to half a parallelogram, whose base is the sum of those two sides, and its altitude the perpendicular distance between them.*

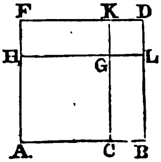


Let ABCD be the trapezoid, having the two sides AB and DC parallel: produce AB until BF is equal to DC, so that AF may be the sum of the two parallel sides; also produce DC, and let the three lines CK, HB, GF be parallel to AD; then will AFGD be a parallelogram of the same altitude with the trapezoid ABCD, having its base AF equal to the sum of the parallel sides of the trapezoid. It now remains to show that the trapezoid ABCD is equivalent to half the parallelogram AFGD. Since parallelograms of equal bases and altitudes are equivalent

(B. II, Prop. I, Cor. 2), we have the parallelogram DK equivalent to HF; also since the figure KH is a parallelogram, the triangle KBC is equal to the triangle BHC (B. I, Prop. XXVII); therefore the line BC divides the parallelogram AG into two equal parts, and the trapezoid ABCD is equivalent to half the parallelogram AG.

### PROPOSITION V.

THEOREM. *The square of the sum of two lines is greater than the sum of their squares, by twice the rectangle of the said lines; or, the square of a whole line is equal to the squares of its two parts, together with twice the rectangle of those parts.*



Let the line AB be the sum of any two lines AC, CB; then will the square of AB be equal to the sum of the squares of AC and CB, together with twice the rectangle AC.CB. That is,

$$AB^2 = AC^2 + CB^2 + 2\,AC.CB.$$

For, let ABDF be the square on the sum or whole line AB, and ACGH the square on the part AC: produce CG and HG to meet the other sides at K and L. From the lines CK, HL, which are equal, being each equal to a side of the square AB or BD (B. I, Prop. XXVII), take the parts CG, HG, which are also equal, being the sides of the square AG, and there remains GK equal to GL, which are also equal to LD and KD, being the opposite sides of the parallelogram GLDK; hence the figure KL is equilateral, and it has all its angles right (B. I, Prop. XXVII, Cor. 1); it is therefore a square on the line GL, or the square of its equal line CB. Also the figures FG, GB are equal to two rectangles under AC and CB, because HG is equal to AC, and GK or GL equal to CB. But the whole square AD is made up of four figures, namely, the two squares AG, GD, and the two equal rectangles FG, GB; that is, the square of AB is equal to the sum of the squares of AC and CB, together with twice the rectangle AC.CB.

*Cor.* If AC is equal to CB, we shall have $AB^2 = 4AC^2$; that is, the square of a line is equal to four times the square of half the line.

(37.) The algebraic formula $(a+b)^2 = a^2 + 2ab + b^2$, is equivalent to the foregoing proposition.

Thus, let $AC = a$; $CB = b$; and consequently $AB = a + b$.

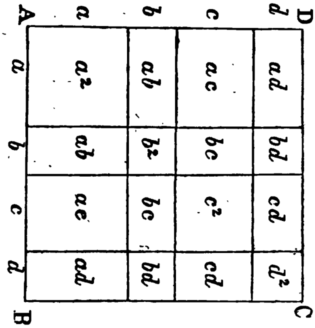
As an example in numbers, suppose $a = 6$; $b = 4$; and we find

$(6+4)^2 = 10^2 = 6^2 + 2.6\times4 + 4^2 = 36 + 48 + 16 = 100$.

(38.) This theorem may be generalized, so as to apply in the case of a line which is the sum of any number of parts.

Let the line AB be equal to the sum of the four lines $a, b, c, d$; then will the square upon AB be made up of $a^2$, $b^2$, $c^2$, $d^2$, $2ab$, $2ac$, $2ad$, $2bc$, $2bd$, $2cd$.

This may be extended to the case where AB is composed of any number of parts, as follows, to wit:

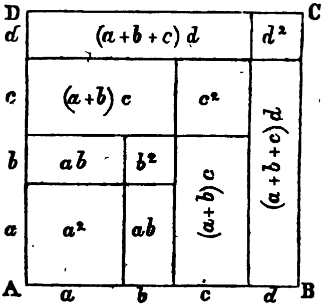


*The square of the sum of any number of lines, is equal to the sum of their squares increased by twice the rectangle of every two.*

(39.) Another method of denoting the square of the sum of any number of lines, is as follows :

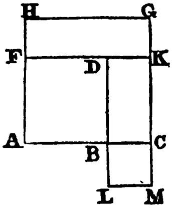
*The square of the sum of any number of lines is equal to the square of the first line, plus twice the rectangle of the first line into the second; plus the square of the second, plus twice the rectangle contained by the sum of the first two into the third; plus the square of the third, plus twice the rectangle of the sum of the first three into the fourth; plus the square of the fourth, and so on.*

This becomes obvious by simply inspecting the following diagram :



### PROPOSITION VI.

**Theorem.** *The square of the difference of two lines is less than the sum of their squares, by twice the rectangle of the said lines.*



Let AC, BC be any two lines, and AB their difference; then will the square of AB be less than the sum of the squares of AC, BC, by twice the rectangle of AC and BC. Or,

$$AB^2=AC^2+BC^2-2\,AC.BC.$$

For, let ABDF be the square on the difference AB, and ACGH the square on the line AC.. Produce FD to K; also produce DB and KC, and draw LM, making BM the square of the line BC. Now it is obvious that the square AD is less than the two squares AG, BM, by the two rectangles FG, DM; but HG is equal to the one line AC, and FH or GK is equal to the other line BC; consequently the rectangle FG, contained under FH and HG, is equal to the rectangle of AC and BC. Again, GK being equal to CM, by adding the common part KC, the whole KM will be equal to the whole GC, or equal to AC; and consequently the figure DM is equal to the rectangle of AC and BC. Hence the two figures FG, DM, are each rectangles of the two lines AC, BC; and consequently the square of AB is less than the sum of the squares of AC, BC, by twice the rectangle AC.BC.

(40.) The algebraic formula $(a - b)^2 = a^2 - 2\,ab + b^2$, is equivalent to this proposition.

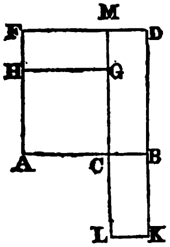
Thus, let AC $= a$; BC $= b$; and consequently AB $= a - b$.

As an example in numbers, let $a = 10$; $b = 6$; and we find

$$(10 - 6)^2 = 4^2 = 10^2 - 2.10 \times 6 + 6^2 = 100 - 120 + 36 = 16.$$

## PROPOSITION VII.

**Theorem.** *The rectangle under the sum and difference of two lines, is equal to the difference of the squares of those lines.*



Let AB, AC be any two unequal lines; then will the difference of the squares of AB, AC, be equal to a rectangle under their sum and difference. That is,

$$AB^2-AC^2=(AB+AC)\,(AB-AC).$$

For, let ABDF be the square of AB, and ACGH the square of AC. Produce DB till BK be equal to AC; draw KL parallel to AB or FD, and produce GC both ways to L and M. Then the difference of the two squares AD, AG, is evidently the two rectangles FG, MB. But the rectangles FG, BL are equal; for FM and BK are each equal to AC, and HF is equal to CB, being equal to the difference between AB and AC, or their equals AF and AH. Therefore the two rectangles FG, MB are equal to the two BL, BM, or to the whole MK; and consequently MK is equal to the difference of the squares AD, AG. But MK is a rectangle contained by DK the sum of AB and AC, and MD the difference of AB and AC. Therefore the difference of the squares of AB and AC is equal to the rectangle under their sum and difference.

(41.) This proposition corresponds with the following algebraic formula : $(a+b)\times(a-b)=a^2-b^2$.

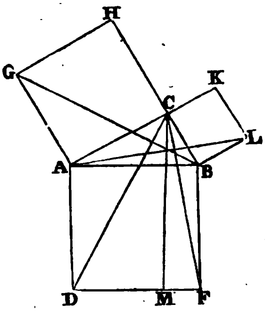
Thus, let $AB=a$; $AC=b$.

If $a=9$, and $b=3$, we shall have

$$(9+3)\times(9-3)=12\times6=9^2-3^2=81-9=72.$$

### PROPOSITION VIII.

**Theorem.** *In any right-angled triangle, the square of the hypothenuse is equal to the sum of the squares of the other two sides.*



Let ABC be a right-angled triangle, having the right angle C; then will the square of the hypothenuse AB be equal to the sum of the squares of the other two sides AC, CB. Or,

$$AB^2 = AC^2 + BC^2.$$

For, on AB describe the square AF, and on AC, CB, the squares AH, BK; then draw CM parallel to AD or BF, and join AL, BG, CD, CF. Now, because the line AC meets the two CH, CB, so as to make two right-angles, these two form one straight line HB (B. I, Prop. 1, Cor. 1); and because the angle GAC is equal to the angle DAB, being each a right angle, or the angle of a square; if to each of these angles we add the common angle BAC, then will the whole angle or sum GAB be equal to the whole angle or sum CAD. But the line GA is equal to the line AC, and the line AB to the line AD, being sides of the same square; so that the two sides GA, AB, and their included angle GAB, are equal to the two sides CA, AD, and the contained angle CAD, each to each.

Therefore the whole triangle AGB is equal to the whole triangle ACD (B. I, Prop. III). But the square AH is double the triangle AGB, on the same base GA, and between the same parallels GA, HB (B. II, Prop. III, Cor.): in like manner the parallelogram AM is double the triangle ACD, on the same base AD, and between the same parallels AD, CM; and since the doubles of equal things are equal (Ax. VI), it follows that the square AH is equal to the parallelogram AM.

In like manner, the other square BK may be shown to be equal to the parallelogram BM. Consequently the two squares AH and BK together are equal to the two parallelograms AM and BM together, or to the whole square AF. That is, the sum of the squares on the two lesser sides is equal to the square on the hypothenuse.

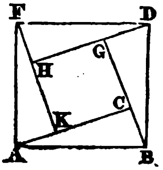
*Cor.* 1. Hence the square of either of the two lesser sides is equal to the difference of the squares of the hypothenuse and the other side (Ax. III); or equal to the rectangle contained by the sum and difference of the hypothenuse and other side (B. II, Prop. VII).

*Cor.* 2. Hence, also, if two right-angled triangles have two sides of the one equal to two corresponding sides of the other, their third sides will be equal, and the triangles themselves equal.

*Cor.* 3. The square on the diagonal of a square is double the square itself.

*Cor.* 4. The sum of the squares of the sides of a rectangle is equal to the sum of the squares of the diagonals.

(42.) This theorem may be demonstrated in the following manner:



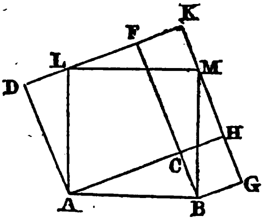
Let ABC be a right-angled triangle, right-angled at C; then will the square upon the hypothenuse AB be equal to the sum of the squares upon the sides AC and BC.

Upon AB construct the square ABDF, so that the triangle ABC may be included within it. Produce the shorter side BC, and from the angle D draw to it the perpendicular DG; which, being produced, draw from the angle F the line FH perpendicular to DH, and produce FH until it meet the longer side of the triangle at the point K. We have thus divided the square ABDF into five portions, namely, four triangles each equal to the original triangle ABC, and the square KCGH, the side of which is equal to KC the difference between AC and BC; consequently the area of this square is

$$(AC - BC)^2 = AC^2 - 2\,AC.BC + BC^2. \qquad \text{[B. II, Prop. vi.]}$$

The area of the triangle ABC is equal to ½AC.BC; therefore the area of the four triangles is equal to 2 AC.BC, which, added to the area of the internal square, gives $AC^2 + BC^2$ for the area of the entire square ABDF; but the area of this square is denoted by $AB^2$, and therefore $AB^2 = AC^2 + BC^2$.

(43.) The following is another simple demonstration:



Let ABC be the right-angled triangle, having the right angle at C. Upon AC and BC, construct the squares AF and BH; produce DF and GH until they meet at K; draw AL and BM each perpendicular to AB, and join LM.

The angles LAB and DAC are each right: taking from each the angle LAC, we have the remaining angles CAB and DAL equal; the angles ACB and ADL are also equal, each being a right angle. We also have DA equal to AC, being sides of the same square; therefore the two triangles ACB and ADL are equal (B. I, Prop. IV). In a similar manner, it may be shown that the triangle BGM is equal to the triangle ACB : hence AL, AB, BM are all equal; and since the angle BAL is right, the figure ALMB is a square. Again, since the sides KL, LM are respectively parallel to CA, AB, and lie in the same direction, the angle KLM is equal to CAB (B. I, Prop. XXII) : for a similar reason, the angle KML is equal to CBA, and the side LM is equal to the side AB; hence the triangle LKM is equal to the triangle ACB (B. I, Prop. IV). Now, since the three triangles ADL, LKM, MGB, are each equal to the triangle ACB, they are equal to each other; consequently the square on AB is equal to the whole space ADKGB diminished by three times the triangle ACB. Again, the rectangle CFKH, contained by CF and CH, which are respectively equal to AC and CB, is double the triangle ACB (B. II, Prop. III) : hence the squares AF and BH are together also equal to the whole space ADKGB diminished by three times the triangle ACB, and consequently these two squares are equivalent to the square upon AB.

(44.) The algebraic condition $(p^2+q^2)^2=(p^2-q^2)^2+(2pq)^2$, enables us to find numerical values for the sides of triangles which shall be right-angled.

Thus, $p^2+q^2=$ hypothenuse; $p^2-q^2$ and $2pq=$ the sides.

If $p=2$, and $q=1$, we shall find

$$p^2+q^2=5=\text{hypothenuse};$$

$$\left.\begin{matrix} p^2-q^2=3 \\ 2pq \quad =4 \end{matrix}\right\}=\text{the sides};$$

and $5^2=3^2+4^2$.

Again, if $p=3$ and $q=1$, we find

$$p^2+q^2=10=\text{hypothenuse};$$

$$\left.\begin{matrix} p^2-q^2=\ 8 \\ 2pq \quad =\ 6 \end{matrix}\right\}=\text{the sides};$$

and $10^2=6^2+8^2$. The sides of this second triangle are double the sides of the first.

(45.) From this we know, that if a triangle is constructed, having 6, 8, and 10 for its sides, it must be right-angled. This relation was formerly very extensively employed by carpenters, in framing by the old *scribe rule.* They referred to it under the name of "*six, eight* and *ten.*" When they wished to ascertain whether the angle formed by the meeting of two sticks of timber was a right angle, they measured from the angular point on the one stick 6 feet, and on the other 8 feet; then if the *ten-foot pole* would reach across from the one point to the other, the angle was right.

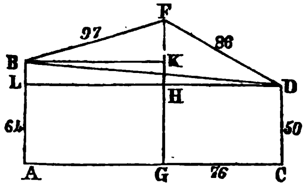
(46.) There is a rule used among carpenters for cutting braces, so as to give 17 inches in length for every 12 inches of *run* and *rise;* that is, they proceed on the supposition that the diagonal of a square 12 inches on a side, is 17 inches. Now this is a little too great; since $12^2 + 12^2 = 288$, while $17^2 = 289$. Notwithstanding this small error, it is found, in the case of short braces, and when the timber is not very firm, to answer very well in practice.

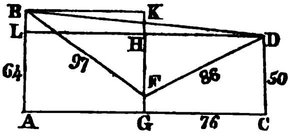
(47.) We may remark, in this place, that our algebraic formula $(p^2 + q^2)^2 = (p^2 - q^2)^2 + (2pq)^2$, readily indicates that it is impossible to find the diagonal of a square in a rational expression; for, in the case of a square, we must have $p^2 - q^2 = 2pq$, or, which is the same thing, $p^2 - 2pq = q^2$. Adding $q^2$ to each side of this equation, we have $p^2 - 2pq + q^2 = 2q^2$. Extracting the square root of each member, we find $p - q = q\sqrt{2}$, where the right-hand member is irrational so long as $q$ is rational.

(48.) PROBLEM. *There are two columns, in the ruins of Persepolis, left standing upright : one is* 64 *feet above the plane, the other* 50 *feet. In a direct line between these, stands an ancient statue, the head of which is* 97 *feet from the top of the higher column, and* 86 *feet from the top of the lower column. The distance from the base of the statue to the base of the lower column is* 76 *feet. Required the distance between the tops of the columns?*

*Solution.* Let AB, CD (next page) represent the columns standing on the horizontal plane, denoted by AC; let G denote the base, and F the head of the statue; then we shall have

AB = 64 feet, CD = 50, BF = 97, DF = 86, and CG = 76.





Join BD, and through B and D draw lines parallel to the horizontal line AC, meeting GF or GF produced at the points K and H : produce DH till it meets AB at L.

$DF^2 - DH^2 = FH^2$; and since $DH = CG$, we have

$FH^2 = 86^2 - 76^2$, and consequently $FH = 40{\cdot}249$ nearly.

Now, $FK = 40{\cdot}249 + 50 - 64 = 26{\cdot}249$; [First figure.]

$FK = 40{\cdot}249 - 50 + 64 = 54{\cdot}249$. [Second figure.]

$BK^2 = BF^2 - FK^2 = 97^2 - (26{\cdot}249)^2$; [First figure.]

$BK^2 = BF^2 - FK^2 = 97^2 - (54{\cdot}249)^2$. [Second figure.]

Hence $BK = 93{\cdot}381$ nearly, or $80{\cdot}412$ nearly.

$DL = BK + CG = 93{\cdot}381 + 76 = 169{\cdot}381$; [First figure.]

$DL = BK + CG = 80{\cdot}412 + 76 = 156{\cdot}412$. [Second figure.]

$BD^2 = BL^2 + DL^2 = 14^2 + (169{\cdot}381)^2$, or $14^2 + (156{\cdot}412)^2$.

Hence $BD = 169{\cdot}96$ feet nearly; [First figure.]

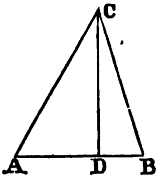
or $BD = 157{\cdot}04$ feet nearly. [Second figure.]

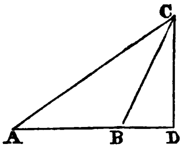
From the above operation, it will be seen that this problem has two real and determinate answers.

(49.) We might multiply examples under this head, to almost any extent desired. Indeed, the preceding theorem is regarded as the most fruitful one in Geometry, conducting as it does to the solution of so many important problems.

## PROPOSITION IX.

THEOREM. *In any triangle, the difference of the squares of the two sides is equal to the difference of the squares of the segments of the base, or of the two lines or distances included between the extremities of the base and the perpendicular.*





Let ABC be any triangle, having CD perpendicular to AB; then will the difference of the squares of AC, BC, be equal to the difference of the squares of AD, BD; that is,

$$AC^2-BC^2 = AD^2-BD^2.$$

For, since (B. III, Prop. VIII) $AC^2 = AD^2 + DC^2$, and $BC^2 = BD^2 + DC^2$, the differences of these are equal; that is, $AC^2 - BC^2 = AD^2 - BD^2$.

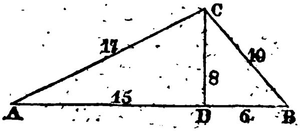
*Cor.* The rectangle of the sum and difference of the two sides of any triangle, is equal to the rectangle of the sum and difference of the distances between the perpendicular and the two extremes of the base, or equal to the rectangle of the base and the difference or the sum of the segments, according as the perpendicular falls within or without the triangle. That is,

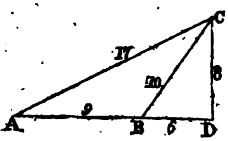
$$(AC+BC)\ (AC-BC)=(AD+BD)\ (AD-BD):$$

or, $(AC+BC)\ (AC-BC)=AB\ (AD-BD)$; [1st figure.]

and $(AC+BC)\ (AC-BC)=AB\ (AD+BD)$. [2d figure.]

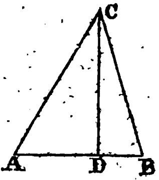
(50.) As an illustration, suppose the length of the lines to be as denoted in the diagrams, we find in each triangle $17^2 - 10^2 = 15^2 - 6^2$.

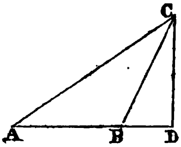




(51.) From this proposition, we may readily deduce a method of determining the area of a triangle when the three sides are known.

Thus, denote the side AB by $b$, the side AC by $c$, and the side BC by $d$.





By this notation, the two last equations on page 60 will become

$$(c + d)(c - d) = b(\text{AD} - \text{BD}); \qquad \text{[1st figure.]}$$
$$(c + d)(c - d) = b(\text{AD} + \text{BD}). \qquad \text{[2d figure.]}$$

Hence, in the first triangle,

$$\text{AD} - \text{BD} = \frac{(c + d)(c - d)}{b} = \frac{c^2 - d^2}{b};$$

and, in the second triangle,

$$\text{AD} + \text{BD} = \frac{(c + d)(c - d)}{b} = \frac{c^2 - d^2}{b}.$$

$$\text{AD} + \text{BD} = b; \qquad \text{[1st figure.]}$$
$$\text{AD} - \text{CD} = b. \qquad \text{[2d figure ]}$$

Now since half the difference of two quantities added to half their sum gives the greater, we have, in both cases,

$$\frac{c^2 - d^2}{2b} + \frac{b}{2} = \text{AD, the greater segment.}$$

Again, $CD = \sqrt{AC^2 - AD^2}$; [B. II, Prop. VIII.]
that is,

$$CD = \sqrt{c^2 - \left(\frac{c^2 - d^2 + b^2}{2b}\right)^2} = \frac{\sqrt{4b^2c^2 - (c^2 - d^2 + b^2)^2}}{2b}.$$

Since the difference of two squares is equal to the product of the sum of the two roots into their difference, we have

$$CD = \frac{\sqrt{(b^2 + 2bc + c^2 - d^2)\,(d^2 - b^2 + 2bc - c^2)}}{2b}$$

$$= \frac{\sqrt{[(b + c)^2 - d^2].[d^2 - (b - c)^2]}}{2b}$$

$$= \frac{\sqrt{(b+c+d)\,(b+c-d)\,(b-c+d)\,(-b+c+d)}}{2b}.$$

Multiplying this perpendicular by half the base, we have, for the area,

$$\frac{\sqrt{(b+c+d)\,(b+c-d)\,(b-c+d)\,(-b+c+d)}}{4},$$

or $$\left\{\frac{b+c+d}{2} \times \frac{b+c-d}{2} \times \frac{b-c+d}{2} \times \frac{-b+c+d}{2}\right\}^{\frac{1}{2}}.$$

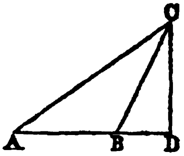
Hence we may find the area of a triangle, when the three sides are known, by this

### RULE.

*Take half the sum of the three sides, and from this half sum subtract each side separately; then take the square root of the continued product of the half sum and the three remainders, and it will be the area.*

## PROPOSITION X.

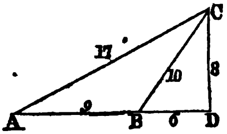
THEOREM. *In any obtuse-angled triangle, the square of the side subtending the obtuse angle is greater than the sum of the squares of the other two sides, by twice the rectangle of the base and the distance of the perpendicular from the obtuse angle.*



Let ABC be a triangle, obtuse-angled at B, and CD perpendicular to AB; then will the square of AC be greater than the squares of AB and BC, by twice the rectangle of AB and BD; that is, $AC^2 = AB^2 + BC^2 + 2\,AB.BD$.

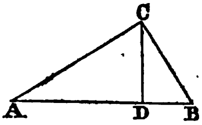
For, $AD^2 = AB^2 + BD^2 + 2\,AB.BD$, [B. II, Prop. v.]
and $AD^2 + CD^2 = AB^2 + BD^2 + CD^2 + 2\,AB.BD$; [Ax. II.]
but $AD^2 + CD^2 = AC^2$, and $BD^2 + CD^2 = BC^2$, [Prop. VIII.]
and therefore $AC^2 = AB^2 + BC^2 + 2\,AB.BD$.

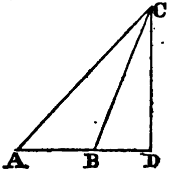
(52.) The above principle may be illustrated by aid of the annexed diagram: $17^2 = 9^2 + 10^2 + 2.9 \times 6$.



### PROPOSITION XI.

THEOREM. *In any triangle, the square of the side subtending an acute angle is less than the sum of the squares of the base and the other side, by twice the rectangle of the base and the distance of the perpendicular from the acute angle.*



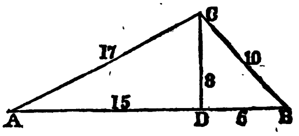


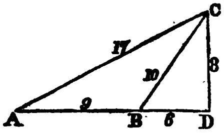
Let ABC be a triangle having the angle A acute, and CD perpendicular to AB; then will the square of BC be less than the squares of AB, AC, by twice the rectangle of AB, AD. That is,

$$BC^2 + 2\,AD.AB = AB^2 + AC^2.$$

For, $BD^2 = AD^2 + AB^2 - 2\,AD.AB$, [B. II, Prop. VI.]
and $BD^2 + DC^2 = AD^2 + DC^2 + AB^2 - 2\,AD.AB$; [Ax. II.]
therefore $BC^2 = AC^2 + AB^2 - 2\,AD.AB$. [B. II, Prop. VIII.]

(53.) This will become obvious in numbers, by the aid of the following diagrams:

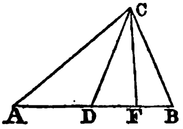




Thus, $10^2 = 21^2 + 17^2 - 2.21 \times 15$; [1st figure.]
$10^2 = 9^2 + 17^2 - 2.\ 9 \times 15.$ [2d figure.]

### PROPOSITION XII.

**Theorem.** *In any triangle, the double of the square of a line drawn from the vertex to the middle of the base, together with double the square of the half base, is equal to the sum of the squares of the other two sides.*



Let ABC be a triangle, and CD the line drawn from the vertex to the middle of the base AB, dividing it into two equal parts AD, DB; then will the sum of the squares of AC, CB, be equal to twice the sum of the squares of CD, AD; or,

$$AC^2 + CB^2 = 2\,CD^2 + 2\,AD^2.$$

For, $AC^2 = CD^2 + AD^2 + 2\,AD.DF$, [B. II, Prop. x.]
and $BC^2 = CD^2 + BD^2 - 2\,AD.DF$; [B. II, Prop. xi.]
hence $AC^2 + BC^2 = 2\,CD^2 + AD^2 + BD^2 = 2\,CD^2 + 2\,AD^2$.

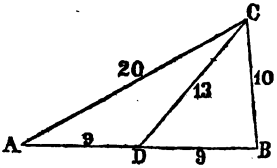
(54.) The numerical lengths of the sides of a triangle, and of the line drawn from the vertex to the middle of the base or opposite side, may be obtained by these algebraic formulæ:

$$\left.\begin{matrix} 2(r^2+1) \\ 2p(r^2+1) \end{matrix}\right\} = \text{sides};$$

$$2(p+1)r^2 - 4(p-1)r - 2(p+1) = \text{base};$$

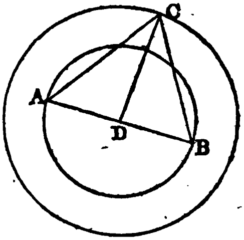
and $(p-1)r^2 + 2(p+1)r - (p-1) =$ line drawn from the vertex to the middle of the base. In which we must take $r > p$.

As a particular case, let $p = 2$, $r = 3$; and we find 20 and 40 for the sides, 36 for the base, and 26 for the line bisecting the base. If we take half of each, we shall have 10 and 20 for the sides, 18 for the base, and 13 for the bisecting line, as in the annexed diagram.



$$AC^2 + BC^2 = 2.CD^2 + 2.AD^2; \quad 20^2 + 10^2 = 2.13^2 + 2.9^2.$$

(55.) THEOREM. *If two concentric circles be described, then, from whatever point in the circumference of the one, lines be drawn to the extremity of any diameter of the other, the sum of their squares will always be the same.*



Let D be the common centre of the two circles; then if from any point C in the circumference of the larger circle, lines be drawn to the extremities of any diameter of the smaller circle, the sum of the squares of these lines will be constant.

For, joining C and D, we have, by the above proposition,

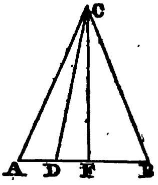
$$CA^2 + CB^2 = 2\,CD^2 + 2\,AD^2.$$

But CD and AD are each constant, being radii of the two circles; therefore $CA^2 + CB^2$ must always amount to the same constant value.

In a similar way it may be shown that the sum of the squares of the two lines, drawn from any point in the circumference of the smaller circle, to the extremities of any diameter of the larger circle, is equal to the same constant quantity, namely, twice the sum of the squares of the two radii.

### PROPOSITION XIII.

THEOREM. *In an isosceles triangle, the square of a line drawn from the vertex to any point in the base, together with the rectangle of the segments of the base, is equal to the square of one of the equal sides of the triangle.*



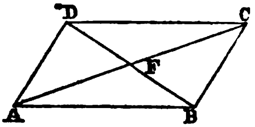
Let ABC be the isosceles triangle, and CD a line drawn from the vertex to any point D in the base; then will the square of AC be equal to the square of CD, together with the rectangle of AD, DB. That is, $AC^2 = CD^2 + AD.DB$.

For, $AC^2 - CD^2 = AF^2 - DF^2 = AD.DB$; [Prop. IX & VII.]

therefore $AC^2 = CD^2 + AD.DB$. [Ax. II.]

### PROPOSITION XIV.

THEOREM. *In any parallelogram, the two diagonals bisect each other; and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.*



Let ABCD be a parallelogram, whose diagonals intersect each other at F; then will AF equal FC, and BF equal FD; and the sum of the squares of AC and BD will be equal to the sum of the squares of AB, BC, CD, DA. That is,

$$AF = FC, \text{ and } BF = FD;$$

$$\text{and } AC^2+BD^2=AB^2+BC^2+CD^2+DA^2.$$

For, the triangles AFB, DFC are identical; since the two lines AC, BD, meeting the parallels AB, DC, make the angle BAF equal to the angle DCF, and the angle ABF equal to the angle CDF (B. I, Prop. XVII): also the side AB is equal to the side DC (B. I, Prop. XXVII); therefore those two triangles are identical (B. I, Prop. IV), and have their corresponding sides equal, namely, $AF=FC$, and $BF=FD$.

Again, since AC and BD are bisected at F, we have (B. II, Prop. XII), $AD^2+DC^2=2\,AF^2+2\,FD^2$, and

$$AB^2+BC^2=2\,AF^2+2\,BF^2; \text{ hence (Ax. II)},$$

$$AB^2+BC^2+CD^2+DA^2=4\,AF^2+4\,FD^2=AC^2+BD^2.$$

[B. II, Prop. V, Cor.].

*Cor.* If $AD=DC$, or the parallelogram be a rhombus, then $AD^2=AF^2+FD^2$; hence the diagonals intersect at right angles (B. II, Prop. VIII).

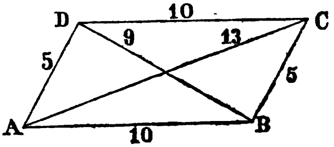
(56.) The numerical lengths of the sides and diagonals of a parallelogram may be found by the following algebraic expressions :

$$\left.\begin{array}{l} r^2+1 \\ p(r^2+1) \end{array}\right\} = \text{sides};$$

$$\left.\begin{array}{l} (p+1)r^2-2\,(p-1)r-(p+1) \\ (p-1)r^2+2\,(p+1)r-(p-1) \end{array}\right\} = \text{diagonals}.$$

These formulæ will always hold good, if we take $r>p$.

As a particular case, let $p=2$, $r=3$. Using these values, we find 10 and 20 for the sides, and 18 and 26 for the diagonals. If we take the half of each, we shall have 5 and 10 for the sides, and 9 and 13 for the diagonals, as in the annexed figure.



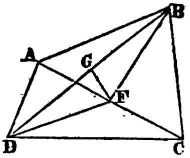
$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2.$$

$$10^2 + \quad 5^2 + \; 10^2 + \quad 5^2 = \; 13^2 + \quad 9^2 = 250.$$

(57.) This theorem may be generalized as follows :

*The squares of the diagonals of a trapezium are together less than the sum of the squares of the four sides, by four times the square of the line joining the points of bisection of the diagonals.*

Let ABCD be a trapezium, whose diagonals AC, BD are bisected in F, G : join FG. The sum of the squares of AC and BD is less than the sum of the squares of the four sides, by four times the square of FG.



Since AC the base of the triangle ABC is bisected by the line BF, we have (B. II, Prop. XII)

$$AB^2+BC^2 = 2\,AF^2+ 2\,BF^2;\ \text{and, for a similar reason,}$$

$$CD^2+DA^2 = 2\,AF^2+ 2\,DF^2;\ \text{therefore}$$

$$AB^2+BC^2+CD^2+DA^2 = 4\,AF^2+ 2\,BF^2+ 2\,DF^2.$$

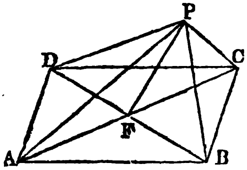
But $4\,AF^2 = AC^2$, [B. II, Prop. V, Cor.],

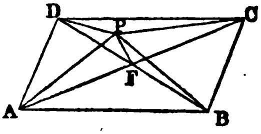
and $2\,BF^2+ 2\,DF^2 = 4\,BG^2+ 4\,FG^2$; [B. II, Prop. XII],

therefore $AB^2+BC^2+CD^2+DA^2 = AC^2+ 4\,BG^2+ 4\,FG^2$

$$= AC^2+ BD^2+ 4\,FG^2.$$

(58.) **Theorem.** *If, from any point whatever, lines be drawn to the four corners of a parallelogram, twice the sum of their squares will be equivalent to the sum of the squares of the diagonals, increased by eight times the square of the line drawn from the given point to the intersection of the diagonals.*





Let lines be drawn from the point P to the corners of the parallelogram ABCD, and to the intersection F of the diagonals. Then, from the triangle PDB, we have (B. II, Prop. XII)

$$PD^2 + PB^2 = 2\,DF^2 + 2\,PF^2;$$

and from the triangle PAC, we have

$$PA^2 + PC^2 = 2\,AF^2 + 2\,PF^2.$$

Hence $PA^2 + PB^2 + PC^2 + PD^2 = 2\,DF^2 + 2\,AF^2 + 4\,PF^2$;

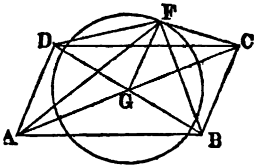
consequently $2\,(PA^2 + PB^2 + PC^2 + PD^2) = BD^2 + AC^2 + 8\,PF^2$.

*Cor.* 1. In case the parallelogram is a rectangle, then, since..... AC = BD, it follows that the sum of the squares of the four lines drawn from the point P to the corners of the rectangle is equivalent to the square of the diagonal, together with four times the square of the line drawn from P to its middle point.

*Cor.* 2. Also, since, in the rectangle, we have DF = AF, it follows, from the two first equations, that the sum of the squares of the lines drawn from P to the opposite corners, is equivalent to the sum of the squares of the two lines drawn from the same point to the other two opposite corners.

*Schol.* If the point P is supposed to be situated at one of the corners of the parallelogram, we shall, as in the preceding general property of the parallelogram, arrive at the relation already established between the squares of the sides and the squares of the diagonals (B. II, Prop. XIV).

(59.) **Theorem.** ***If, from the central point of any parallelogram, as a centre, a circle be described with any radius, the sum of the squares of the four lines drawn from any point in the circumference, to the four corners of the parallelogram, will always amount to the same sum.***

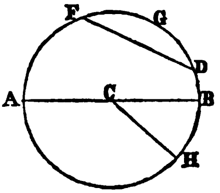


From G the centre of the parallelogram ABCD, let a circumference of a circle be described, having any radius; also from F any point in this circumference, let lines be drawn to the four corners of the parallelogram. Then, (Art. 58), twice the sum of the squares of these four lines will be equal to the sum of the squares of the diagonals, together with eight times the square of the radius FG; but the diagonals and the radius always remain the same, and therefore the squares of the four lines thus drawn will always amount to the same sum.

# BOOK THIRD.

## DEFINITIONS.

1. Any portion of the circumference of a circle is called an *arc*.



2. The straight line joining the extremities of an arc is called a *chord*. The chord is said to *subtend* the arc.

3. The portion of the circle included by an arc and its chord, is called a *segment*.

Thus the space FGDF, included by the arc FGD and the chord FD, is a segment; so also is the space included by the same chord and the arc FAHBD.

4. The portion included between two radii and the intercepted arc, is called a *sector*.

The space BCH is a sector.

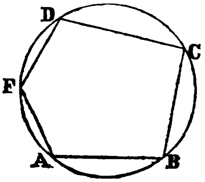
5. When a straight line touches the circumference in only one point, it is called a *tangent;* and the common point of the line and circumference is called the *point of contact*.

6. One circle *touches* another, when their circumferences have only one point common.

7. A line is *inscribed* in a circle, when its extremities are in the circumference.

8. An angle is inscribed in a circle, when its sides are inscribed.

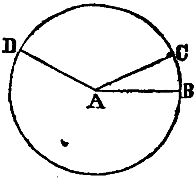
9. A polygon is inscribed in a circle when its sides are inscribed; and under the same circumstances, the circle is said to *circumscribe* the polygon.



Thus AB is an inscribed line, ABC an inscribed angle, and the figure ABCDF an inscribed polygon.

10. A circle is inscribed in a polygon when its circumference touches each side, and the polygon is said to be *circumscribed about the circle.*

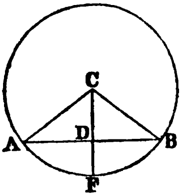
11. By *an angle in a segment* of a circle, is to be understood an angle whose vertex is in the arc, and whose sides intercept the chord of said arc; and by *an angle at the centre,* is meant one whose vertex is at the centre. In both cases, the angles are said to be *subtended* by the chords or arcs which their sides include.



12. The circumference of a circle may be described by causing the extremity B of the line AB to revolve about the other extremity A, which remains fixed. In this revolution, while the line AB passes over the angular space BAC, the extremity B passes over the arc BC; and while the line passes over the angular space CAD, its extremity describes the arc CD; and so for other angles. Hence the angles at the centre are measured by the arcs included between their sides.

## PROPOSITION I.

**Theorem.** *If a line, drawn through or from the centre of a circle, bisect a chord, it will be perpendicular to the chord; or, if the line be perpendicular to the chord, it will bisect both the chord and the arc of the chord.*



Let AB be any chord in a circle, and CD a line drawn from the centre C to the chord; then, if the chord be bisected at the point D, CD will be perpendicular to AB.

Drawing the two radii CA, CB, and comparing the two triangles ACD, BCD, which have CA equal to CB (Def. XXIII), and CD common, as also AD equal to DB by hypothesis, we find that the three sides of the one are equal to the three sides of the other; therefore the angle CDA is equal to the angle CDB (B. I, Prop. viii), and each of these angles is a right angle, and CD is perpendicular to AB (Def. X).

Again, if CD be perpendicular to AB, then will the chord AB be bisected at the point D, or have AD equal to DB; and the arc AFB will be bisected at the point F, or have AF equal to FB.

For, drawing CA, CB as before, we have the triangle CAB isosceles, and consequently the angle CAD is equal to CBD (B. I, Prop. v); we also have the angles CDA and CDB equal, each being a right angle (Def. X); therefore the third angles of these triangles are equal (B. I, Prop. xxiv, Cor. 1), and, having the side CD common, they must also have the side AD equal to the side DB (B. I, Prop. IV).

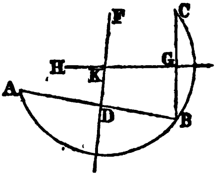
Also, since the angle ACF is equal to the angle BCF, the arc AF which measures the former is equal to the arc BF which measures the latter.

*Cor.* Hence a line bisecting any chord at right angles, passes through the centre of the circle.

(60.) By means of this proposition, we may find the centre of a given circle, or of a given arc of a circle.

Let ABC be the arc of a circle : it is required to find its centre.

Draw any two chords AB, BC; bisect them with the perpendiculars DF and GH; then will the point K, in which they intersect, be the centre sought.



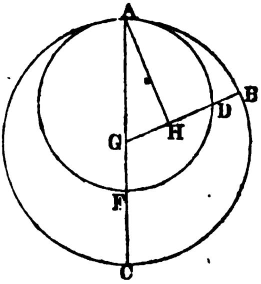
For, by the corollary of the above proposition, each of the perpendiculars DF and GH passes through the centre, and therefore the centre is at the point K.

(61.) It is also obvious that the above operation is equivalent to finding the centre of a circle whose circumference will pass through three given points A, B, C. Hence, by this means, a circumference of a circle may always be made to pass through any three points, not situated in the same straight line.

(62.) The same method is applicable, in case it is required to describe a circle about a triangle.

### PROPOSITION II.

THEOREM. *If two circles, the one placed within the other, touch, the centres of the circles and the point of contact will be in the same straight line.*



Let the two circles ABC, ADF touch one another internally at the point A; then will A and the centres of those circles be all in the same straight line.

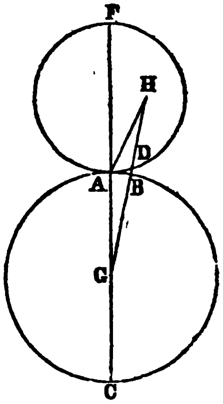
Let G be the centre of the circle ABC, through which draw the diameter AGC; then, if the centre of the other circle can be out of this line AC, let it be supposed at some other point H, through which draw the line GH, cutting the two circles in B and D: join AH.

Now, in the triangle AGH, the sum of the two sides GH, HA is greater than the third side AG (B. I, Prop. XVI), or greater than its equal, the radius GB. From each of these take away the common part GH, and the remainder HA will be greater than the remainder HB. But the point H being supposed the centre of the inner circle, its two radii HA, HD are equal to each other; consequently HD will also be greater than HB.

Again, ADF being the inner circle, HD is necessarily less than HB; so that HD is both greater and less than HB, which is absurd. To remove this absurdity, we must abandon the supposition which led to it, which was that the centre H might be out of the line AGC. Consequently the centre H cannot be out of the line AGC; that is, the the line GH, joining the centres, passes through A the point of contact.

### PROPOSITION III.

**Theorem.** ***If two circles touch one another externally, the centres of the circles and the point of contact will be all in the same straight line.***



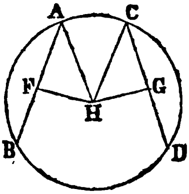
Let the two circles ABC, ADF touch one another externally at the point A; then will the point of contact A, and the centres of the circles, be all in the same straight line.

Let G be the centre of the circle ABC, through which draw the diameter AGC, and produce it to meet the other circle at the point F. Then, if the centre of the other circle ADF can be out of the line CF, let it, if possible, be supposed to be at some other point as H; and draw the lines AH, GBDH, cutting the two circles in B and D.

Then, in the triangle AGH, the sum of the two sides AG, AH is greater than the third side GH (B. I, Prop. xvi). But since G and H are the centres of the two circles, the two radii HA, HD are equal; so also are the two radii GA, GB; hence the sum of AG, AH is equal to the sum of GB, HD. We have just shown that the sum of AG, AH is greater than the distance GH; consequently the sum of GB, HD is greater than GH, which is absurd. Consequently, as in the last proposition, the centre H cannot be out of the line FC.

### PROPOSITION IV.

THEOREM. *Chords in a circle, which are equally distant from the centre, are equal to each other. Conversely, if chords in the same circle are equal to each other, they will be equally distant from the centre.*



Let AB, CD be any two chords equally distant from the centre H; then will these two chords be equal to each other.

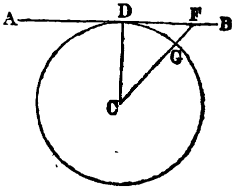
Draw the two radii HA, HC, and the two perpendiculars HF, HG, which are the equal distances of the chords from the centre H. Then the two right-angled triangles HAF, HCG have the sides HA and HC equal (being radii), the side HF equal to HG, and the angle HFA equal to the angle HGC, each being a right angle; therefore those two triangles are equal (B. II, Prop. VIII, Cor. 2), and consequently AF is equal to CG. But AB is the double of AF, and CD the double of CG (B. III, Prop. I); therefore AB is equal to CD (Ax. VI).

Conversely, if the chord AB is equal to the chord CD, then will their distances from the centre, HF, HG, be equal to each other.

For, since AB is equal to CD, we must have AF the half of AB equal to CG the half of CD. We also have the radii HA, HC equal, and the angle HFA equal to the angle HGC, each being a right angle; therefore the triangle HAF is equal to the triangle HCG (B. II, Prop. VIII, Cor. 2), and consequently HF is equal to HG.

## PROPOSITION V.

THEOREM. *A line perpendicular to a radius, at its extremity, is tangent to the circumference.*

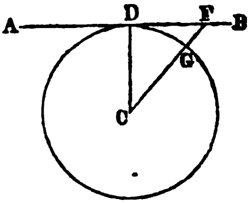


Let the line ADB be perpendicular to the radius CD at its extremity; then will AB touch the circumference at the point D only.

From any other point, as F, in the line AB, draw FGC to the centre, cutting the circle at the point G. Then because the angle CDF of the triangle CDF is a right angle, the angle DFC is acute (B. I, Prop. xxiv, Cor. 1), and consequently less than the angle CDF. But the longer side is always opposite the greater angle (B. I, Prop. xv); therefore the side CF is longer than the side CD, or its equal CG. Hence the point F is without the circle; and the same may be shown of any other point of the line AB, except the point D. Consequently the line AB touches the circumference at only one point, and is therefore tangent to it.

## PROPOSITION VI.

THEOREM. *When a line is tangent to the circumference of a circle, a radius drawn to the point of contact is perpendicular to the tangent.*



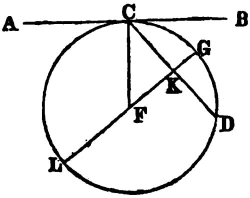
Let the line AB be tangent to the circumference of a circle at the point D; then will the radius CD be perpendicular to the tangent AB.

For, the line AB being wholly without the circumference except at the point D, it follows that any line, as CF, drawn from the centre C to meet the line AB at any point different from D, must have its extremity F without the circumference. Hence the radius CD is the shortest line that can be drawn from the centre to meet the tangent AB, and therefore CD is perpendicular to AB (B. I, Prop. xxvi).

*Cor.* Hence, conversely, a line drawn perpendicular to a tangent at the point of contact, passes through the centre of the circle.

### PROPOSITION VII.

**Theorem.** *The angle formed by a tangent and chord is measured by half the arc of that chord.*



Let AB be a tangent, and CD a chord drawn from the point of contact C; then the angle BCD will be measured by half the arc CGD, and the angle ACD will be measured by half the arc CLD.

Draw the radius FC to the point of contact, and the radius FG perpendicular to the chord CD, meeting it at the point K. Then the radius FG, being perpendicular to the chord CD, bisects the arc CGD (B. III, Prop. i); therefore CG is half the arc CGD.

In the triangle CFK, the angle CKF being a right angle, the sum of the two remaining angles CFK and FCK is equal to a right angle (B. I, Prop. XXIV, Cor.), which is consequently equal to the right angle FCB. From each of these equals, take away the common angle FCK, and there remains the angle CFK equal to the angle BCD. But the angle CFK is measured by the arc CG (B. III, Def. 12), which is half of the arc CGD; therefore the equal angle BCD must also be measured by half the arc CGD.

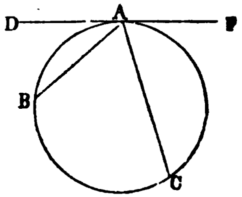
Again, the line LFG, being perpendicular to the chord CD, bisects the arc CLD (B. III, Prop. I); therefore the arc CL is half the arc CLD. Now, the line CF, meeting GL, makes the sum of the two angles CFG, CFL equal to two right angles (B. I, Prop. I), and the line CD makes with AB the two angles DCB, DCA together equal to two right angles. If from these equals we take the equals CFG, DCB, we shall have the remainders CFL, DCA equal. Now the former of these, CFL, being an angle at the centre, is measured by the arc CL, which is half the arc CLD; therefore the angle DCA is also measured by half the arc CLD.

*Cor.* 1. The sum of two right angles is measured by half the circumference; for the two angles BCD, ACD, which together make two right angles, are respectively measured by the arcs CG, CL, which make half the circumference, LG being a diameter.

*Cor.* 2. Hence also one right angle must have for its measure a quarter of the circumference, or 90 degrees.

### PROPOSITION VIII.

**THEOREM.** *An angle at the circumference of a circle is measured by half the arc that subtends it.*



Let BAC be an angle at the circumference : it has for its measure, half the arc BC which subtends it.

For, suppose the tangent DF to pass through the angular point A. Then, the angle DAC being measured by half the arc ABC, and the angle DAB by half the arc AB (B. III, Prop. VII), it follows that the difference of those angles is measured by half the difference of the said arcs ; that is, the angle BAC is measured by half the arc BC upon which it stands.

*Cor.* 1. All angles in the same segment of a circle, or standing upon the same arc, are equal to each other ; for each is measured by half the same arc.

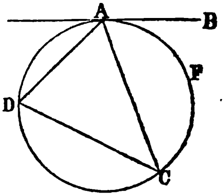
*Cor.* 2. An angle at the centre of a circle is double the angle at the circumference, when both stand on the same arc. For the angle at the centre is (B. III, Def. 12) measured by the whole arc on which it stands, and the angle at the circumference is measured by half the arc on which it stands ; consequently the angle at the centre is double the angle at the circumference.

*Cor.* 3. An angle in a semicircle is a right angle ; for it is measured by half a semicircumference, or by a quadrant, which is the measure of a right angle (B. III, Prop. VII, Cor. 2).

*Cor.* 4. The sum of any two opposite angles of a quadrilateral inscribed in a circle, is equal to two right angles; for, as each inscribed angle is measured by half the arc on which it stands, it follows that the two opposite angles of an inscribed quadrilateral, together, must be measured by half the entire circumference, which is the measure of two right angles.

### PROPOSITION IX.

THEOREM. *The angle formed by a tangent to a circle, and a chord drawn from the point of contact, is equal to the angle in the alternate segment.*

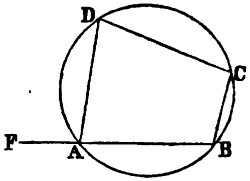


If AB be a tangent, AC a chord, and D any angle in the alternate segment ADC ; then will the angle D be equal to the angle BAC made by the tangent and the chord of the arc AFC.

For the angle D at the circumference is measured by half the arc AFC (B. III, Prop. VIII), and the angle BAC made by the tangent and chord is also measured by half the same arc AFC (B. III, Prop. VII); therefore these two angles are equal.

### PROPOSITION X.

THEOREM. *If any side of a quadrilateral, inscribed in a circle, be produced out, the outward angle will be equal to the inward and opposite angle.*

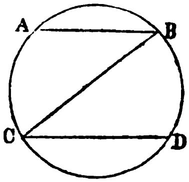


If the side BA of the quadrilateral ABCD, inscribed in a circle, be produced to F, the outward angle DAF will be equal to the inward and opposite angle C.

For, the sum of the two adjacent angles DAF and DAB is equal to two right angles (B. I, Prop. I), and the sum of the two opposite angles C and DAB is also equal to two right angles (B. III, Prop. VIII, Cor. 4); therefore the two angles DAF and DAB are together equal to the sum of DAB and C. From each, take the common angle DAB, and we have the remainders DAF and C equal to each other.

## PROPOSITION XI.

**Theorem.** *Any two parallel chords intercept equal arcs.*

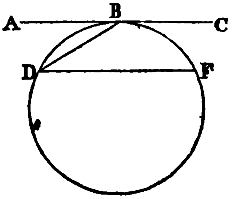


Let the chords AB, CD be parallel; then will the arcs AC, BD be equal.

Draw the line BC; then, because AB and CD are parallel, the alternate angles B and C are equal (B. I, Prop. XVII). But the angle B at the circumference is measured by half the arc AC (B. III, Prop. VIII), and the equal angle C at the circumference is measured by half the arc BD; therefore the arc AC is equal to the arc BD.

### PROPOSITION XII.

**Theorem.** *When a tangent and chord are parallel to each other, they intercept equal arcs.*

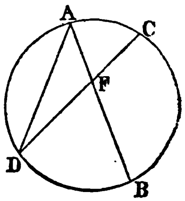


Let the tangent ABC be parallel to the chord DF ; then will the arc BD equal BF.

Draw the chord BD. Then, since the lines AC, DF are parallel, the alternate angles DBA and BDF are equal ( B. I, Prop. XVII). But the angle DBA, formed by a tangent and chord, is measured by half the arc BD (B. III, Prop. VII) ; and its equal angle BDF, being at the circumference, is measured by half the arc BF (B. III, Prop. VIII) ; therefore the arc BD is equal to BF.

### PROPOSITION XIII.

**Theorem.** *The angle formed within a circle, by the intersection of two chords, is measured by half the sum of the two intercepted arcs.*



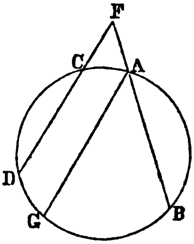
Let the two chords AB, CD intersect at the point F ; then will the angle AFC, or its equal BFD, be measured by half the sum of the arcs AC and BD.

Draw the chord AD ; then will the angle AFC be equal to the sum of the two angles FAD, FDA (B. I, Prop. XXV). The angle FAD is measured by half the arc DB (B. III, Prop. VIII) ; for the same reason, the angle FDA is measured by half the arc AC. Therefore the angle AFC, which is equal to the sum of the two angles FAD, FDA, is measured by half the sum of the two arcs DB and AC.

In the same way, by drawing the chord BD, it may be shown that the angle AFD, or its equal BFC, is measured by half the sum of the two arcs AD and BC.

### PROPOSITION XIV.

THEOREM. *The angle formed, out of a circle, by two secants, is measured by half the difference of the intercepted arcs.*



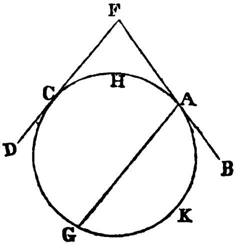
Let the angle F be formed by the two secants FAB, FCD; this angle will be measured by half the difference of the two arcs BD and AC intercepted by the two secants.

Draw the chord AG parallel to FD. Then, because the lines FD and AG are parallel, and BF meets them, the exterior angle BAG is equal to the interior and opposite angle BFD (B. I, Prop. xx). But the angle BAG, being at the circumference, is measured by half the arc BG (B. III, Prop. VIII), or by half the difference of the arcs BD and GD; therefore the angle BFD is also measured by half the difference of the arcs BD and GD.

Again, because the chords CD and AG are parallel, the arc AC is equal to GD (B. III, Prop. XI); therefore the difference of the two arcs BD and AC is equal to the difference of the two BD and GD, and consequently the angle BFD is measured by half the difference of the arcs BD and AC.

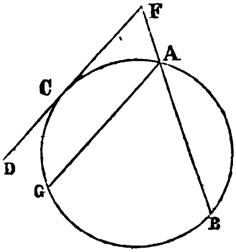
### PROPOSITION XV.

THEOREM. *The angle formed by two tangents is measured by half the difference of the two intercepted arcs.*



Let FB, FD be two tangents to a circle at the points A, C; then will the angle F be measured by half the difference of the two arcs AGC, CHA.

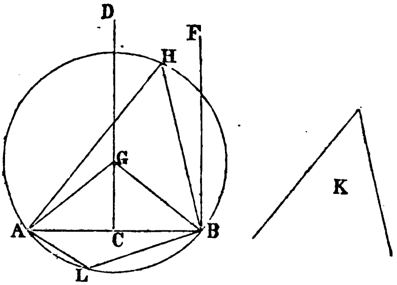
Draw the chord AG parallel to FD. Then, because the lines AG and FD are parallel, and BF meets them, the exterior angle BAG is equal to the interior and opposite angle BFD (B. I, Prop. xx). But the angle BAG, formed by a tangent and chord, is measured by half the arc AKG (B. III, Prop. vii); therefore the equal angle F will also be measured by half the same arc AKG, or half the difference of the arcs AGC and GC, or half the difference of the two AGC and CHA (B. III, Prop. xi).



*Cor.* In like manner, it is proved that the angle F, formed by a tangent FCD and a secant FAB, is measured by half the difference of the two intercepted arcs BGC and CA.

## PROPOSITION XVI.

**Problem.** *On a given line as a chord, to describe a segment of a circle, capable of containing an angle of a given magnitude.*



Let AB be the given line, and K the given angle.

Draw CD, bisecting AB at right angles; also, at B, draw BF perpendicular to AB; then draw BG, making the angle FBG equal to the given angle K (B. I, Prop. x).

Joining GA, and comparing the two triangles GCB and GCA, we have GC common, and the side CB equal to CA; also the angle GCB equal to the angle GCA, each being a right angle; therefore GB is equal to GA (B. I, Prop. III). Hence, if, with G as a centre, a circumference be described, passing through the point B, it will also pass through the point A, and the segment AHB will be the one required.

Since the angle AGB at the centre is measured by the arc ALB, and the angle AHB at the circumference is measured by half the same arc (B. III, Prop. VIII), it follows that the angle AHB is half the angle AGB; therefore the angle AHB, inscribed in the segment AHB, is equal to the angle BGC, the half of BGA; but the angle BGC is equal to the alternate angle FBG (B. I, Prop. xvii), which was made equal to the given angle K; therefore the angle AHB, in the segment AHB, is equal to the given angle K.

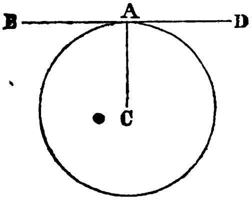
*Schol.* If an angle ALB be inscribed in the other segment of the circle, it will, with the angle AHB, be equal to the sum of two right angles (B. III, Prop. viii, Cor. 4). Hence, when the given angle K is a right angle, the line AB will be a diameter; and when the angle K is acute, then will the angle ALB be obtuse.

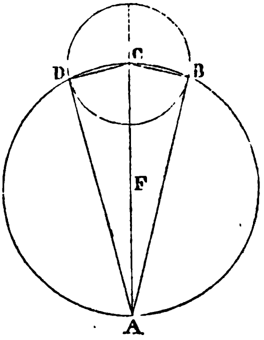
When the given angle K is obtuse, if we find a segment capable of containing the angle which remains when K is taken from two right angles, then will the opposite segment contain angles equal to the given angle K.

### PROPOSITION XVII.

PROBLEM. *Through a given point, to draw a tangent to a given circumference.*

If the given point A is situated in the circumference, draw the radius CA; and through the point A draw BD perpendicular to CA, and it will be the tangent required (B. III, Prop. v).





If the point A is situated without the circumference, join AC, bisect it at F, and with F as a centre, and a radius equal to FA or FC, describe a circumference cutting the given circumference at the points B and D. Draw AB and AD, and they will each be tangents as required.

For, joining CB and CD, we know that the angles ABC, ADC are right angles, being inscribed in a semicircle (B. III, Prop. VIII, Cor. 3); hence the lines AB, AD are tangent to the given circumference (B. III, Prop. VI).

In this case, the two tangents are of the same length. For, in the two right-angled triangles ABC, ADC, we have AC a common hypothenuse, and the sides BC, CD equal, each being a radius of the same circle; therefore those triangles are equal (B. II, Prop. VIII, Cor. 2). Consequently AB is equal to AD.

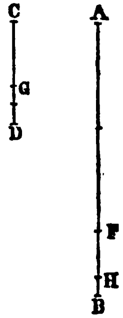
# BOOK FOURTH.

## DEFINITIONS.

1. For the doctrine of ratios and proportions, we will refer the student to the method explained in the Algebra.

2. There is this difference between geometrical ratios of magnitudes, and ratios of numbers : All numbers are *commensurable;* that is, their ratio can be accurately expressed : but many magnitudes are *incommensurable;* that is, their ratio can be expressed only by approximation; which approximation may, however, be carried to any extent we desire. Such is the ratio of the circumference of a circle to its diameter, the diagonal of a square to its side, etc. Hence many have deemed the arithmetical method not sufficiently general to apply to geometry. This would be a safe inference, were it necessary in all cases to assign the specific ratio between the two terms compared. But this is not the case. Such ratios themselves may be unknown, indeterminate, or irrational, and still their equality or inequality may be as completely determined by the arithmetical method as by the more lengthy method of the Greek geometers.

(63.) **Problem.** *To find a common measure of two given lines, and consequently their numerical ratio.*



Let AB and CD be the given lines.

From the greater AB cut off parts equal to the lesser line CD, as many times as possible; for example, twice, with the remainder FB.

From the line CD cut off parts equal to FB, as many times as possible; for example, once, with the remainder GD.

From the first remainder FB cut off parts equal to the remainder GD, as many times as possible; for example, once, with the remainder HB.

From the second remainder GD cut off parts equal to the third remainder HB, as many times as possible; for example, twice, without a remainder.

The last remainder HB will be a common measure of the given lines.

If we regard HB as a unit, GD will be 2, and

$$FB = FH + HB = GD + HB = 3;$$
$$CD = CG + GD = FB + GD = 3 + 2 = 5;$$
$$AB = AF + FB = 2\,CD + FB = 10 + 3 = 13.$$

Therefore the line AB is to the line CD as 13 to 5.

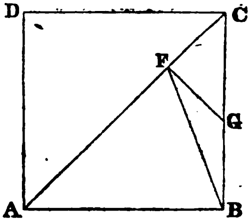
If AB is taken for the unit, CD will be $\frac{5}{13}$; but if CD be taken as the unit, AB will be $\frac{13}{5}$.

If AB is $\frac{1}{2}$ of a yard, then CD will be $\frac{5}{13}$ of $\frac{1}{2}$ a yard, or $\frac{5}{26}$ of a yard.

Again, if CD is $\frac{2}{3}$ of a foot, then AB will be $\frac{13}{5}$ of $\frac{2}{3}$ of a foot $= \frac{26}{15}$ of a foot; and so on for other comparisons.

(64.) As a case in which the magnitudes are incommensurable, we will take the following

**Problem.** *Find the ratio of the diagonal of a square to its side.*



Let ABCD be the square, and AC its diagonal.

Cutting off AF from the diagonal equal to AB a side of the square, we have the remainder CF which must be compared with CB.

If we join FB, and draw FG perpendicular to AC, the triangle BGF will be isosceles.

For, the angle ABG = AFG, each being a right angle; and since the triangle ABF is isosceles, the angle ABF = AFB. Therefore, subtracting the angle ABF from ABG, the remainder FBG will equal the angle BFG, found by subtracting the angle AFB from AFG. Consequently the triangle BGF is isosceles, and BG = FG; but, since AC is the diagonal of a square, the angle FCG is half a right angle; but CFG is a right angle, and consequently FGC is also half a right angle, and CG is the diagonal of a square whose side is CF.

Hence, after CF has been taken once from CB, it remains to take CF from CG, that is, to compare the side of a square with its diagonal, which is the very question we set out with, and of course we shall find precisely the same difficulty in the next step of the process; so that, continue as far as we please, we shall never arrive at a term in which there will be no remainder. Therefore there is no common measure of the diagonal and side of a square.

If the side of a square be represented by 1, then arithmetically the diagonal will be $\sqrt{2}$, and this value can be found only approximately.

The student may say, that in this case, we have two numbers which have not even a unit for their common measure. I reply, that $\sqrt{2}$ is not a number: it is simply an expression for a ratio, the arithmetical value of which can only be found approximately.

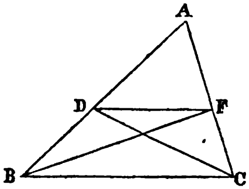
3. Similar figures are those which have the angles of the one equal to the angles of the other, each to each, and the sides about the equal angles proportional.

4. The *perimeter*, or *contour* of a figure, is the sum of all its sides, or the length of the bounding line.

5. Two magnitudes are said to be *identical*, when they are equal in all their parts, and are also capable of coinciding in every part of their extent.

### PROPOSITION I.

THEOREM. *If a line be drawn parallel to one side of a triangle, cutting the other two sides, these sides will be divided into proportional parts.*



Let DF be parallel to the side BC of the triangle ABC; then will AD : DB : : AF : FC.

For, draw BF and CD; then the triangles DBF, FCD are equal to each other, because they have the same base DF, and are between the same parallels DF, BC (B. II, Prop. I, Cor. 1). But the two triangles ADF, BDF, on the bases AD, DB, have the same altitude; and the two triangles ADF, CDF, on the bases AF, FC, have also the same altitude; and because triangles of the same altitude are to each other as their bases, therefore

$$ADF : BDF :: AD : DB;$$
$$ADF : CDF :: AF : FC.$$

But BDF = CDF; consequently, by equality of ratios, we have $AD : DB :: AF : FC$.

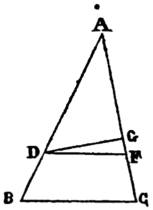
In a similar manner, the theorem is proved when the sides of the triangle are cut in prolongation beyond either the vertex or the base.

*Cor.* Hence, also, the whole lines AB, AC are proportional to their corresponding proportional segments. Thus, since $AD : DB :: AF : FC$, we have by composition $AD+DB : AD :: AF+FC : AF$, or

$$AB : AD :: AC : AF;\text{ and}$$
$$AD+DB : DB :: AF+FC : FC,\text{ or}$$
$$AB : DB :: AC : FC.$$

### PROPOSITION II.

**Theorem.** *If two sides of a triangle are cut proportionally by a straight line, this line will be parallel to the third side.*



In the triangle ABC, let the line DF be drawn, so that

$$AD : DB :: AF : FC;$$

then will DF be parallel to BC.

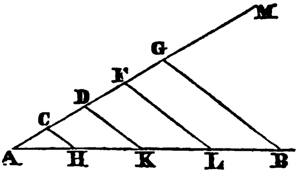
For if DF is not parallel to BC, suppose that from the point D the line DG be drawn parallel to BC. Then we have (B. IV, Prop. 1)

$AD : DB :: AG : GC$. But, by hypothesis, we have $AD : DB :: AF : FC$; therefore we must have $AG : GC :: AF : FC$, or

$AG : AF :: GC : FC$, which is an impossible result, since the antecedent of the first couplet is less than its consequent, while the antecedent of the second couplet is greater than its consequent. Hence the line drawn from D parallel to BC cannot differ from the line DF; that is, DF is parallel to BC.

### PROPOSITION III.

**Problem.** *To divide a given line* AB *into any given number of equal parts.*

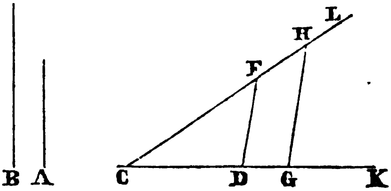


Draw any straight line AM making an angle with AB, and upon it set off equal portions of any convenient length AC, CD, DF, FG, as many times as the number of parts into which the line AB is to be divided; in the present case, four times. Join GB, and draw CH, DK, FL parallel to GB; and then will the line be divided at the points H, K, L, as required.

For, by reason of parallels, we have (B. IV, Prop. I) AC : CD : DF.... : : AH : HK : KL.... ; and as the former are all equal, the latter will also be equal.

### PROPOSITION IV.

**Problem.** *To find a third proportional to two given lines* A *and* B.

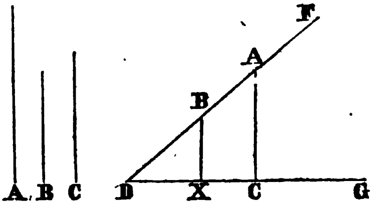


Draw the two indefinite lines KC and LC, making any angle with each other. On CL and CK take CF and CG each equal to B, and on CK take CD equal to A; then join DF, and through G draw GH parallel to DF, and CH will be a third proportional to the lines A and B.

For, since DF is parallel to GH, we have (B. IV, Prop. I) CD : CF :: CG : CH, or, which is the same thing, A : B :: B : AH.

### PROPOSITION V.

Problem. *To find a fourth proportional to three given lines* A, B, C.



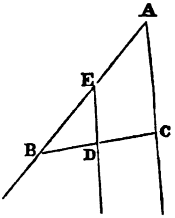
Draw the two indefinite lines DF, DG, making any angle with each other. Upon DF take DA = A, and DB = B; upon DG take DC = C; join AC, and through the point B draw BX parallel to AC. DX will be the fourth proportional required; for since BX is parallel to AC, we have (B. IV, Prop. I)

DA : DB :: DC : DX; that is,

A : B :: C : DX.

### PROPOSITION VI.

PROBLEM. *Through a given point, in a given angle, to draw a line so that the segments comprehended between the point and the two sides of the angle shall be equal.*



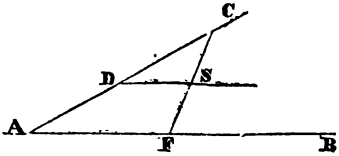
Let BAC be the given angle, and D the given point.

Through the point D draw DE parallel to AC; make EB equal to AE, and through B and D draw BC, and it will be the line required.

For, ED being parallel to AC, we have

BE : EA : : BD : DC; [B. IV, Prop. I.]

but BE=EA, and therefore BD=DC.

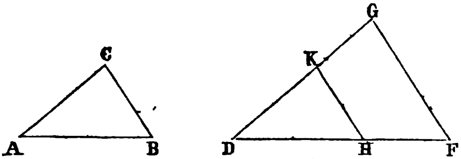


(65.) Let AB and AC be two roads, and suppose S to be a well of water : it is required to pass a new road by S, so that the distance SF may equal SC.

*Solution.* Through S draw SD parallel to AB; make DC equal to DA; draw the line CS, and produce it to F, and it will be the position of the required road (B. IV, Prop. VI).

### PROPOSITION VII.

**Theorem.** *Equiangular triangles are similar, or have their like sides proportional.*



Let ABC, DFG be two equiangular triangles, having the angle A equal to the angle D, the angle B equal to the angle F, and the angle C equal to the angle G; then will AB : AC : : DF : DG.

For, make DH=AB and DK=AC, and join KH; then the two triangles ABC, DHK, having the two sides AB, AC respectively equal to DH, DK, and the contained angles A and D also equal, are identical, or equal in all respects (B. I, Prop. III); hence the angle B is equal to the angle DHK, and the angle C is equal to the angle DKH. But the angles B and C are equal respectively to the angles F and G by hypothesis; therefore also the angles DHK and DKH are equal respectively to the angles F and G (Ax. I), and consequently the line HK is parallel to the side FG (B. I, Prop. XX, Cor. 1).

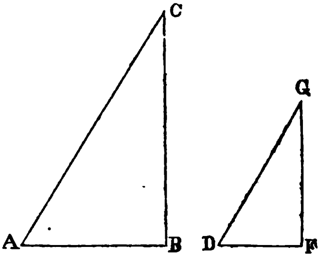
Hence, then, in the triangle DFG, the line HK, being parallel to the side FG, it divides the other two sides into proportional parts, so that

DH : DK : : DF : DG. [B. IV, Prop. I.]

But DH and DK are respectively equal to AB and AC; therefore also AB : AC : : DF : DG.

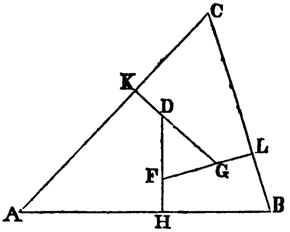
## PROPOSITION VIII.

THEOREM. *Two triangles, which have their homologous sides parallel or perpendicular, are similar.*



If, in the two triangles ABC, DFG, the side AB is parallel to DF, AC parallel to DG, and BC parallel to FG, then will these triangles be similar.

For, since AC, AB are respectively parallel to DG, DF, the angle at A is equal to the angle at D (B. I, Prop. xxii). For a similar reason, the angle B equals the angle F, and the angle C equals the angle G; therefore the two triangles are similar (B. IV, Prop. vii).

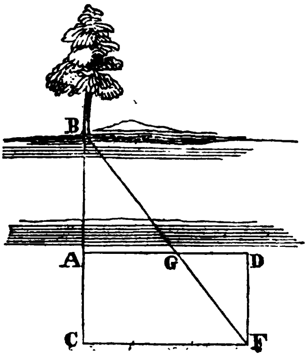


Again, in the two triangles ABC, DFG, suppose DF to be perpendicular to AB, FG perpendicular to BC, and DG perpendicular to AC; then will the triangle DFG be similar to the triangle ABC.

When the triangles are so situated that one is wholly within the other, produce the sides of the innermost triangle until they meet the sides of the outer triangle to which they are perpendicular. Then, in the quadrilateral AHDK, the angles AHD and AKD are each a right angle; consequently the sum of the two angles HAK and HDK is equal to two right angles (B. I, Prop. xxiv, Cor. 2). But the sum of the two angles HDK and FDG is also equal to two right angles (B. I, Prop. i); therefore the sum of the two angles HAK, HDK is equal to the sum of the two angles FDG, HDK; and taking the angle HDK from each of these equals, we have the angles HAK and FDG equal. In a similar manner it may be shown that the angles B and DFG are equal, also the angles C and DGF; consequently these two triangles are similar.

When one of the triangles is situated without the other, or partly without and partly within, we may construct a new triangle having its sides parallel with the sides of one of the given triangles, and lying wholly within the other; and since this new triangle is situated wholly within, it must be similar to the triangle in which it is situated. Consequently, by the first part of this proposition, the triangle which has its sides parallel to the sides of the new triangle must also be similar to the other given triangle.

(66.) Let it be required to find the width AB of a river, without crossing it.



Take a position C in a direct line with A and B; also take any point, as D, near the bank of the river. Then, having driven stakes at A, C and D, stretch a line from C to A, and thence to D: mark that point of the line which touches the stake at A. Remove the line, and exchange ends; that is, fasten to the stake D the end which was first made fast at C, and fasten upon C the end D; then, taking hold of the line at the point which was in contact with A, stretch the line so that it shall take the position CFD. The figure ADFC is evidently a parallelogram, since, by construction, the opposite sides are equal. On the line AD, take the point G in a direct line with FB. The two triangles FDG and BAG are similar, since the angles ABG and DFG are alternate in reference to the parallels BC, DF, and are therefore equal (B. I, Prop. XVII); also the angle AGB is equal to FGD, being opposite angles (B. I, Prop. II). These two triangles, having equal angles, are similar, and we have

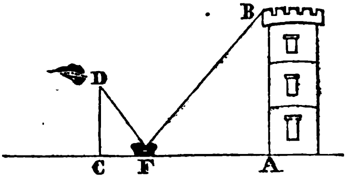
GD : DF : : GA : AB. [B. IV, Prop. VII.]

In this proportion, the first, second and third terms being known, we can find the fourth term, which is the distance sought.

(67.) **Problem.** ***To find the height of a perpendicular object, by means of an artificial horizon.***

An artificial horizon is a horizontal surface of any substance capable of reflecting light uniformly, as the surface of mercury, ink, etc. If soft treacle be placed in a saucer, it will form a very good artificial horizon.

Place the artificial horizon at F, and the eye at D, so that the top of the object may be seen by reflection in F. Measure the height CD of the eye, and the distances CF and FA; then, since the triangles FCD and FAB are obviously similar, we have

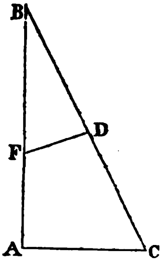


CF : CD : : FA : AB.

(68.) The perpendicular height of an object may be found by means of its shadow, by placing a stick, of known length, perpendicularly in the ground, and measuring the length of its shadow; then make the following proportion:

Shadow of the stick : shadow of the object : : height of the stick : height of the object.

**Thales** is said to have taught the Egyptians how to measure the height of the pyramids, by means of their shadows.



(69.) PROBLEM. *Suppose* AB *to be a tree, standing on the horizontal plane* AC; *it is required to find at what point it must break, so that, by falling, the top may strike the ground at* C (See Art. 25).

Let the height of the tree be denoted by $h$, and the distance AC by $b$; then BC will equal

$\sqrt{b^2 + h^2}$, [B. II, Prop. VIII.]

and consequently CD $= \frac{1}{2}$ BC $= \frac{1}{2}\sqrt{b^2 + h^2}$.

Again, the two triangles BDF and CAB are similar, having the angle B common, and the angle BDF equal to the angle BAC, each being a right angle (B. IV, Prop. VII); therefore

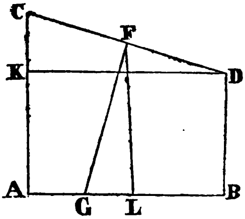
$$BA : BC :: BD : BF,$$

or

$$h : \sqrt{b^2 + h^2} :: \tfrac{1}{2}\sqrt{b^2 + h^2} : BF.$$

$$\text{Consequently } BF = \frac{b^2 + h^2}{2h},$$

$$\text{and } AF = AB - BF = h - \frac{b^2 + h^2}{2h} = \frac{h^2 - b^2}{2h}.$$



(70.) PROBLEM. *Suppose* AC *and* BD *to represent two trees standing on the horizontal plane* AB; *it is required to find a point in this plane, situated on the line* AB, *equally distant from the tops* C *and* D (See Art. 22).

Join CD, and bisect it with the perpendicular FG; then, by Art. 22, G is the point sought.

This point may be found arithmetically as follows:

Draw DK parallel to AB, and FL perpendicular to AB. Let the height of the tree AC be denoted by $h_1$, the height of the tree BD by $h_2$, and the distance AB by $d$; then will KC $=$ AC $-$ BD $= h_1 - h_2$, FL $= \frac{1}{2}(h_1 + h_2)$.

Since the sides of the two triangles DKC, FLG are respectively perpendicular to each other, they are similar (B. IV, Prop. VIII), and we have DK : KC :: FL : LG; or, in symbols,

$$d : h_1 - h_2 :: \tfrac{1}{2}(h_1 + h_2) : LG.$$

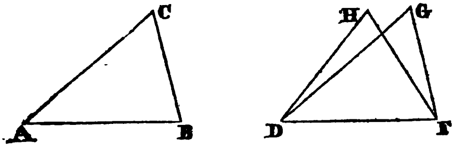
Consequently $LG = \dfrac{h_1^2 - h_2^2}{2d}$;

and $AG = AL - LG = \tfrac{1}{2}d - \dfrac{h_1^2 - h_2^2}{2d} = \dfrac{d^2 - h_1^2 + h_2^2}{2d}$,

$$BG = BL + LG = \tfrac{1}{2}d + \frac{h_1^2 - h_2^2}{2d} = \frac{d^2 + h_1^2 - h_2^2}{2d}.$$

## PROPOSITION IX.

**THEOREM.** *Triangles which have their sides proportional, are equiangular.*



In the two triangles ABC, DFG, if

$$AB : DF :: AC : DG :: BC : FG,$$

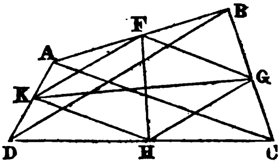
the two triangles will have their corresponding angles equal.

For, if the triangle ABC be not equiangular with the triangle DFG, suppose some other triangle having the same base DF, as DFH, to be equiangular with ABC. But this is impossible; for if the two triangles ABC, DFH were equiangular, their sides would be proportional (B. IV, Prop. VII), and we should have

$AB : DF :: AC : DH$; but, by supposition, $AB : DF :: AC : DG$, and therefore $DH = DG$. We should also have

$AB : DF :: BC : FH$; but, by supposition, $AB : DF :: BC : FG$, and therefore $FH = FG$. Hence the two triangles DFH and DFG are identical, since all the sides of the one are equal to all the sides of the other (B. I, Prop. VIII); which is absurd, since their angles are unequal. Therefore the triangles ABC and DFG must be equiangular.

(71.) THEOREM. *The sum of the squares of the diagonals of a quadrilateral is twice the sum of the squares of the two lines bisecting the opposite sides.*



Let ABCD be a quadrilateral, whose sides are bisected with the lines FH and GK; then will $AC^2 + BD^2 = 2(FH^2 + GK^2)$.

For, draw FG, GH, HK & KF; and then, since BA and BC are divided proportionally, each being halved, the line FG must be parallel to AC (B. IV, Prop. II). In the same way we may show that HK is parallel to AC, and therefore FG and HK are parallel. Furthermore we shall have GH and FK each parallel to BD, and therefore parallel to each other. Consequently the figure FGHK is a parallelogram (Def. XVII).

Again, since the sides of the triangles FAK, BAD are respectively parallel, they are similar, and $BA : FA :: BD : FK$; and since BA is twice FA, it follows that BD is twice FK. In the same way, it may be shown that AC is twice FG. Hence

$$BD^2 = 4FK^2 = 2FK^2 + 2GH^2;$$
$$AC^2 = 4FG^2 = 2FG^2 + 2HK^2.$$

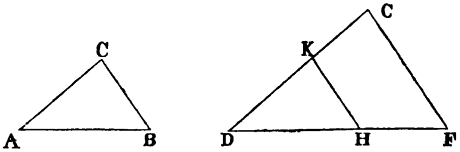
Consequently $AC^2 + BD^2 = 2(FG^2 + GH^2 + HK^2 + KF^2)$
$$= 2(FH^2 + GK^2). \quad \text{[B. II, Prop. XIV.]}$$

*Cor.* 1. From the above demonstration, it follows, that if the points of bisection of the four sides of a trapezium be joined, we shall thus form a parallelogram.

*Cor.* 2. Also the lines drawn to the middle points of the opposite sides of a trapezium mutually bisect each other, since they are the diagonals of the parallelogram mentioned in Cor. 1.

## PROPOSITION X.

THEOREM. *Triangles, which have an angle in the one equal to an angle in the other, and the sides about these angles proportional, are equiangular.*

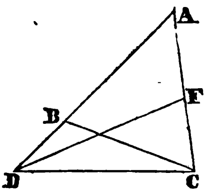


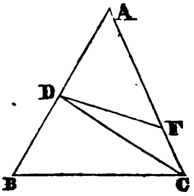
Let ABC, DFG be two triangles, having the angle A equal to the angle D, and the sides AB, AC proportional to the sides DF, DG; then will the triangle ABC be equiangular with the triangle DFG.

For, make DH equal to AB, and DK equal to AC, and join HK. Then the two triangles ABC, DHK, having two sides and the contained angle of the one equal to the two sides and the contained angle of the other, are equal (B. I, Prop. III), and the angle B is equal to the angle DHK, the angle C to the angle DKH. The sides DH, DK, being respectively equal to AB, AC, are proportional to DF, DG; therefore HK is parallel to FG (B. IV, Prop. II). Consequently the angles DHK, DKH are respectively equal to DFG, DGF (B. I, Prop. XX); but the angles DHK and DKH have just been shown to be respectively equal to the angles B and C, and consequently the angles DFG and DGF are respectively equal to the angles B and C.

## PROPOSITION XI.

THEOREM. *Two triangles having an angle in each equal, are to each other as the rectangles of the sides which contain the equal angles.*





Let the two triangles ABC, ADF have the angle A common; then will the

triangle ABC : triangle ADF : : AB.AC : AD.AF.

For, joining D and C, we have, since triangles of the same altitude are to each other as their bases,

triangle ABC : triangle ADC : : AB : AD, and

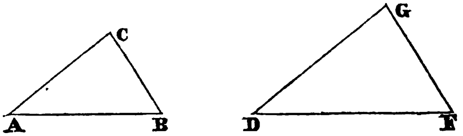
triangle ADC : triangle ADF : : AC : AF.

Multiplying together the corresponding terms of these proportions, and omitting the common term, triangle ADC, which enters into the antecedent and consequent of the first couplet, we have

triangle ABC : triangle ADF : : AB.AC : AD.AF.

### PROPOSITION XII.

THEOREM. *Equiangular or similar triangles are to each other as the squares of their homologous sides.*



Let ABC, DFG be two equiangular triangles, AB and DF being two like sides; then will the triangle ABC be to the triangle DFG, as the square of AB is to the square of DF, or as $AB^2$ to $DF^2$.

For, the triangles being similar, they have their like sides proportional (B. IV, Prop. VII), and are to each other as the rectangles of the like pairs of their sides (B. IV, Prop. XI). Therefore,

$$AB : DF :: AC : DG,$$

and $$AB : DF :: AB : DF;$$

therefore, taking the product of the corresponding terms of these proportions, we have

$$AB^2 : DF^2 :: AB.AC : DF.DG.$$ But the

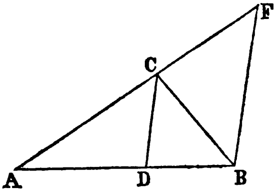
$$\text{triangle } ABC : \text{triangle } DFG :: AB.AC : DF.DG.$$
[B. IV, Prop. XI.]

Therefore $\text{triangle } ABC : \text{triangle } DFG :: AB^2 : DF^2$.

### PROPOSITION XIII.

**Theorem.** *A line which bisects any angle of a triangle, divides the opposite side into two segments, which are proportional to the two other adjacent sides.*

Let the angle ACB of the triangle ABC be bisected by the line CD, making the angle ACD equal to the angle DCB; then will the segment AD be to the segment DB, as the side AC is to the side BC. Or,



AD : DB : : AC : CB.

For, let BF be drawn parallel to CD, meeting AC produced at the point F; then, because the line BC meets the two parallel lines CD, FB, it makes the angle CBF equal to the alternate angle BCD (B. I, Prop. xvii). Again, because AF meets the two parallel lines CD, FB, the exterior angle ACD is equal to the interior and opposite angle CFB (B. I, Prop. xx). Now since the angles BCD and ACD are equal, it follows that the angle CBF is equal to CFB; therefore the triangle CBF is isosceles, and the side CF is equal to CB (B. I, Prop. vii). Now, in the triangle ABF, since CD is parallel to BF, we have

AD : DB : : AC : CF; [B. IV, Prop. i.]

or, substituting CB for its equal CF,

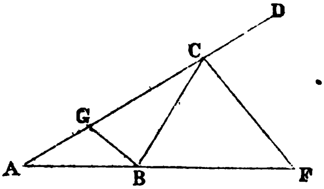
AD : DB : : AC : CB.

(72.) We may also show the following proposition to be true :

THEOREM. *If the exterior angle, formed by producing one of the sides of a triangle, be bisected by a line which meets the opposite side or base produced, then will the sides of the triangle be to each other as the distances from the extremities of the base to the point where the bisecting line meets the base produced.*

In the triangle ABC, let the side AC be produced, forming the exterior angle BCD : let CF bisect this angle, and meet the base produced at F ; then will

AC : BC : : AF : BF.



For, through B draw BG parallel to CF ; then will the angle .... CBG = BCF (B. I, Prop. XVII), also CGB = DCF (B. I, Prop. XX) ; and since, by hypothesis, the angle BCF = DCF, we must have .... CBG = CGB. Therefore the triangle CBG is isosceles, and CG = CB (B. I, Prop. VII).

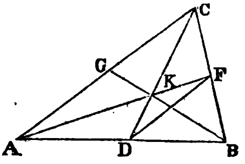
Again, since BG is parallel to FC, we have AC : GC : : AF : BF (B. IV, Prop. I) ; and substituting BC for GC, its equal, we have

AC : BC : : AF : BF.

## PROPOSITION XIV.

THEOREM. *If lines be drawn from the vertices of a triangle to bisect the opposite sides, they will mutually trisect each other.*

In the triangle ABC, let CD, AF, BG be lines drawn from the vertices to bisect the opposite sides ; then will they mutually trisect each other, that is, $DK = \frac{1}{3} CD$ ; $FK = \frac{1}{3} AF$ ; $GK = \frac{1}{3} BG$.



For, if we join DF, it will be parallel to AC (B. IV, Prop. II), and the angles KFD and KAC will be equal, and also the angles KDF and KCA will be equal (B. I, Prop. XVII). Moreover the angle DKF will be equal to AKC (B. I, Prop. II); and therefore the two triangles DKF and AKC, being equiangular, are similar (B. IV, Prop. VII). These triangles being similar, we have

$$DF : AC :: DK : KC;$$

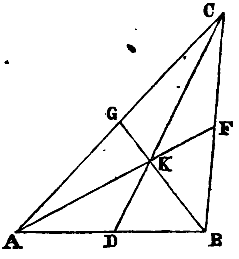
$$DF : AC :: FK : KA.$$

But since DF is drawn bisecting the two sides BA, BC, it is equal to half of AC; therefore we have $DK = \frac{1}{2} KC$, $FK = \frac{1}{2} KA$, which proves that AF and CD trisect each other.

In the same way it may be shown that AF and BG, as well as CD and BG, trisect each other, and consequently the three lines mutually trisect each other.

*Cor.* This proposition shows that the three lines drawn from the vertices of any triangle, so as to bisect the opposite sides, will all pass through the same point.

(73.) THEOREM. *If lines be drawn from the vertices of a triangle, bisecting the opposite sides, these lines will bisect the triangle itself.*

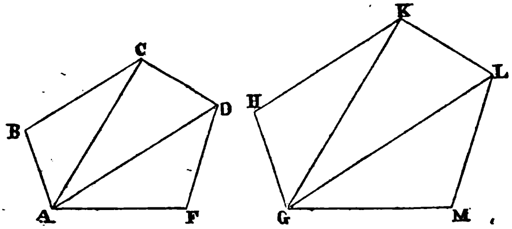


Since CD, drawn from the vertex C, bisects the base AB, it will of necessity bisect every line parallel to this base, drawn between the sides AC, BC; consequently it must bisect the triangle itself. In a similar manner, it may be shown that AF and BG also bisect this triangle. Or, since the two triangles CAD, CDB have equal bases and the same altitude, they are equal (B. II, Prop. II, Cor. 2).

The point K in which all these lines meet, is obviously the *centre of gravity* of the triangle. Hence the centre of gravity of a triangle may be found by drawing a line from any vertex to the middle of the opposite side, and then taking two-thirds of this line from the vertex.

### PROPOSITION XV.

THEOREM. *Two similar polygons are composed of the same number of similar triangles, similarly situated.*



Let ABCDF, GHKLM be two similar polygons.

From any angle A in the polygon ABCDF, draw the diagonals AC, AD; and from the angle G in the other polygon, homologous with A, draw the diagonals GK, GL.

These polygons being similar, the angles ABC, GHK, which are homologous, are equal, and the sides AB, BC must also be proportional to the sides GH, HK (B. IV, Def. 3). Therefore the two triangles ABC, GHK have an angle of the one equal to an angle of the other, and the sides about those angles proportional, and consequently these triangles are similar; and being similar, we have the angle BCA equal to the angle HKG. But since the polygons are similar, the angle BCD is equal to the angle HKL; therefore ACD, which is the difference between the angles BCD and BCA, is equal to the angle GKL, which is the difference between the angles HKL and HKG (Ax. III).

Since the triangles ABC and GHK are similar, we have

BC : HK : : AC : GK ;

and since the polygons are similar, we have

BC : HK : : CD : KL.

Therefore, by equality of ratios, we have

AC : GK : : CD : KL.

Hence the two triangles ACD and GKL have an angle of the one equal to an angle of the other, and the sides about those angles proportional, and consequently the triangles are similar.

In the same manner it might be shown that all the remaining triangles are similar, whatever be the number of sides of the proposed polygons. Therefore two similar polygons are composed of the same number of similar triangles, similarly situated.

*Scholium.* The converse of this proposition is also true: *If two polygons are composed of the same number of similar triangles similarly situated*, *those polygons will be similar.*

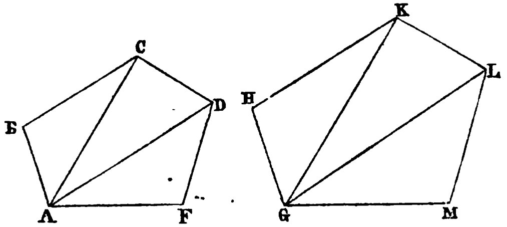
For, the similarity of the respective triangles will give the angles ABC=GHK, BCA=HKG, ACD=GKL, &c. Hence BCD=HKL; likewise CDF=KLM, &c. Moreover we shall have

AB : GH : : BC : HK : : AC : GK : : CD : KL, &c.

Hence the two polygons have their angles equal and their sides proportional, and consequently they are similar.

### PROPOSITION XVI.

THEOREM. *The perimeters of similar polygons are to each other as their homologous sides; and their areas are to each other as the squares of those sides.*



*First.* By the nature of similar polygons, we have

AB : GH : : BC : HK : : CD : KL, &c.

Now the sum of these antecedents AB+BC+CD, &c., which makes the perimeter of the first polygon, is to the sum of their consequents GH + HK + KL, &c., which makes the perimeter of the second polygon, as any one antecedent is to its corresponding consequent, and therefore as AB is to GH.

*Secondly.* Since the triangles ABC, GHK are similar, we have the triangle ABC : GHK : : $AC^2$ : $GK^2$ (B. IV, Prop. XII); also since the triangles ACD and GKL are similar, we have triangle ACD : GKL : : $AC^2$ : $GK^2$; therefore, by reason of the common ratio $AC^2$ : $GK^2$, we have ABC : GHK : : ACD : GKL.

By the same mode of reasoning, we should find

ACD : GKL : : ADF : GLM; and so on, if there were more triangles.

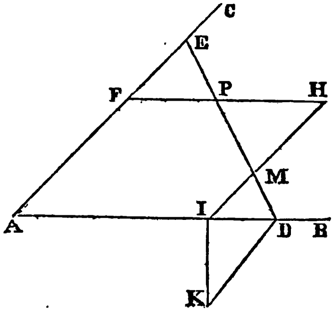
From this series of equal ratios, we conclude

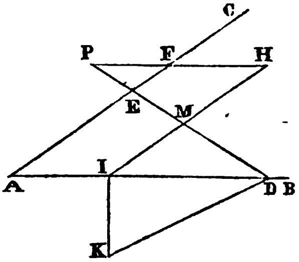
that the sum of the antecedents ABC + ACD + ADF,
or the polygon ABCDF,
is to the sum of the consequents GHK + GKL + GLM,
or the polygon GHKLM,
as any one antecedent ABC,
is to its corresponding consequent GHK,
or as $AB^2$ is to $GH^2$.

Hence the areas of similar polygons are to each other as the squares of their homologous sides.

*Cor.* If three similar rectilinear figures are constructed on the three sides of a right-angled triangle, the figure on the hypothenuse will be equivalent to the sum of the other two; for the three figures are to each other as the squares of their homologous sides, and the square of the hypothenuse is equivalent to the sum of the squares of the other two sides.

(74.) **Problem.** *Through a given point* P, *to draw a straight line* PED *to cut two straight lines* AB, AC *given in position, so that the triangle* ADE *thus formed may be of a given magnitude.*



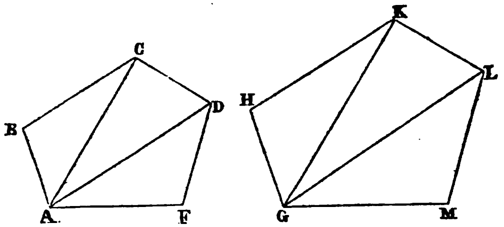


Draw PFH parallel to AB, intersecting AC at F; and upon AF construct the parallelogram AFHI, equal in area to the given area of the triangle. Make IK perpendicular to AI, and equal to FP; and, from the point K to AB, apply KD = PH; then draw DPE, and the problem will be solved.

For, supposing M to be the intersection of DE and IH, it is evident, because of the parallel lines, that the three triangles PHM, PFE and MDI are equiangular. Therefore, all equiangular triangles being in the ratio of the squares of their homologous sides ( B. IV, Prop. XII), and the sum of the squares of FP or IK and DI being equal to the square of PH or KD by construction ( B. II, Prop. VIII), it is evident that the sum of the triangles PFE and DMI is equal to the triangle PHM ( this is obvious also from B. IV, Prop. XVI, Cor.); to which equal quantities in fig. 1, add AFPMI, and we shall have ADE equal to AFHI. But in fig. 2, let PFE be taken from PHM, and there will remain EFHM = DMI; to each of which, adding AIME, we have AFHI = ADE as before.

## PROPOSITION XVII.

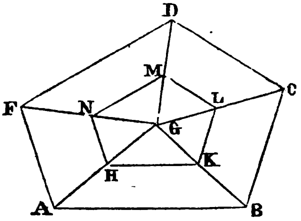
PROBLEM. *On a given line* GH *homologous to a given side* AB *of a given rectilineal figure* ABCDF, *to construct a figure similar to the given one.*

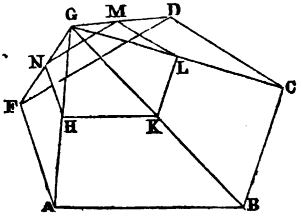


From one of the extremities of the line AB which is homologous to GH, draw lines to all the angles of the figure; draw HK, making the angle GHK equal to the angle ABC (B. I, Prop. x); also draw GK, making the angle HGK equal to the angle BAC; then will the triangle GHK be similar to the triangle ABC (B. IV, Prop. VII). In a similar manner, on GK, homologous with AC, construct the triangle KGL similar to the triangle ACD; also on the side GL, homologous with AD, construct the triangle GLM similar to the triangle ADF. Then will the polygon GHKLM be similar to the polygon ABCDF.

For, these two polygons are composed of the same number of similar triangles similarly situated, and therefore they are similar (B. IV, Prop. xv, Schol.).

(75.) PROBLEM. *To make a polygon similar to a given polygon, and having its perimeter in a given ratio to the perimeter of the given polygon.*





Let ABCDF be the given polygon.

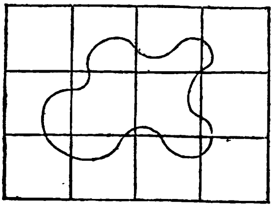
From any point either within or without the polygon, as G, draw lines to all the angles of the polygon. Then take GH, GK, GL, GM, GN, each in the same ratio to the corresponding lines GA, GB, GC, GD, GF, that the perimeter of the required polygon is to be to the perimeter of the given polygon. Join the points H, K, L, M, N, and the polygon HKLMN will be the polygon required.

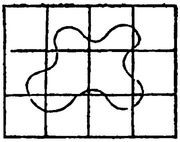
These polygons are evidently similar, for the corresponding sides are parallel (B. IV, Prop. II); and since, in the triangle GAB, HK is parallel to AB, we have HK : AB : : GK : GB; and as the side

KL is parallel to BC, we have GK : GB : : KL : BC. Therefore, by equality of ratios, we have HK : KL : : AB : BC.

In the same way it may be shown that KL : LM : : BC : CD, and so on for the other sides. Therefore these polygons are similar.

(76.) PROBLEM. *To draw a complex figure similar to another figure, on the same or different scales, by means of rectangles.*



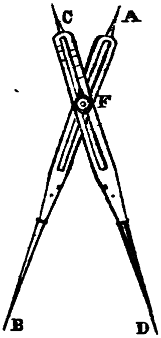


Surround the given figure by a square or a rectangle of convenient size, and divide it by pencil lines, intersecting perpendicularly, into squares or rectangles, as small as may be deemed necessary. Generally, the more irregular the contour of the figure, or the more numerous the sinuosities or subdivisions, the more numerous the rectangles should be.

Then draw another square or rectangle, having its sides either equal to the former, or greater or less in the assigned ratio, and divide this figure into as many squares or rectangles as there are in the original figure. Draw, in every rectangle of the new figure, right lines or curved, to agree with what is contained in the corresponding rectangle of the original figure; and this, if carefully done, will give a correct copy of the complex diagram proposed.

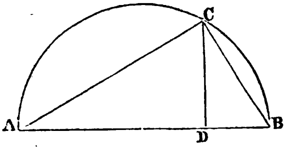
In ornamental needle-work, the same system of copying is practised. The figures to be executed are usually required to be wrought on coarse canvas, the threads of which form a system of squares, such as above described. The original object from which the copy is made, is delineated in proper colors on paper on which a similar system of squares is printed, the color occupying each square being there distinctly expressed; so that the needle-worker sees, at once, what color of silk or worsted it is necessary to make use of for each respective square.

(77.) The Proportional Compass is an instrument much used by artists in the reduction of pictures, etc. It consists of two similar and equal pieces of brass AB and CD, terminated at each end by steel points. F is a pivot, which may be adjusted so as to divide the length of the two legs into any required ratio. Whatever be the ratio of AF to FB, or of CF to FD, the same will be the ratio of the distance AC to BD, whatever may be the extent to which the compass is opened.



### PROPOSITION XVIII.

**Theorem.** *In a right-angled triangle, a perpendicular drawn from the right angle to the hypothenuse is a mean proportional between the segments of the hypothenuse; and each of the sides about the right angle is a mean proportional between the hypothenuse and adjacent segment.*



Let ABC be a right-angled triangle, and CD a perpendicular from the right angle C to the hypothenuse AB; then will

CD be a mean proportional between AD and DB,
AC a mean proportional between AB and AD, and
BC a mean proportional between AB and BD;

or

$$AD : CD :: CD : DB,$$
$$AB : AC :: AC : AD, \quad \text{and}$$
$$AB : BC :: BC : BD.$$

For, the two triangles ABC, ADC, having the right angles at C and D equal, and the angle A common, have their third angles equal, and are equiangular (B. I, Prop. XXIV, Cor. 1). In like manner, the two triangles ABC, BDC, having the right angles at C and D equal, and the angle B common, have their third angles equal, and are equiangular. Hence all the three triangles ABC, ADC, BDC, being equiangular, will have their like sides proportional (B. IV, Prop. VII), viz.

$$AD : CD :: CD : DB,$$

$$AB : AC :: AC : AD, \quad \text{and}$$

$$AB : BC :: BC : BD.$$

*Cor.* 1. Since an angle in a semicircle is a right angle (B. III, Prop. VIII, Cor. 3), it follows, that if, from any point C in a semicircumference, a perpendicular be drawn to the diameter AB, and the two chords CA, CB be drawn to the extremities of the diameter, then are CD, AC, BC the mean proportionals as in this proposition; or

$$CD^2=AD.DB, \quad AC^2=AB.AD, \text{ and } BC^2=AB.BD.$$

*Cor.* 2. Hence $AC^2 : BC^2 :: AD : BD$.

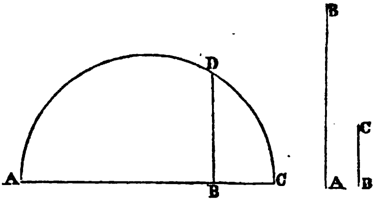
*Cor.* 3. Hence we have another demonstration of Proposition VIII, Book II.

For, since $AC^2=AB.AD$, and $BC^2=AB.BD$; by addition, $AC^2+BC^2 = AB\,(AD+BD) = AB^2$.

### PROPOSITION XIX.

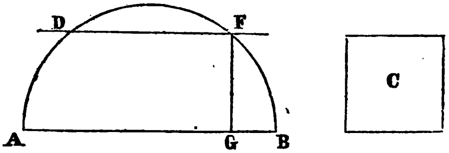
PROBLEM. *To find a mean proportional between two given lines* AB, BC.

Let them be placed in a straight line, as in the figure. On AC as a diameter, describe a semicircumference ADC; and through the point B draw BD perpendicular to AB, meeting the semicircumference at the point D; then will BD be a mean proportional to AB, BC. This follows immediately from B. IV, Prop. XVIII, Cor. 1.



### PROPOSITION XX.

**PROBLEM.** *To construct a rectangle equivalent to a given square, and having the sum of its adjacent sides equal to a given line.*



Let C be the given square, and AB equal to the sum of the sides of the required rectangle.

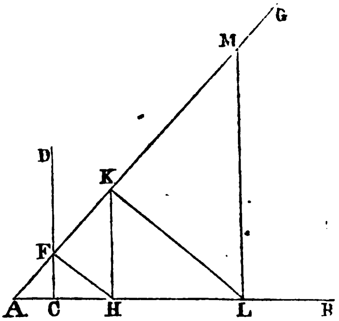
Upon AB as a diameter, describe the semicircumference ADFB; and draw DF parallel to AB, at a distance equal to a side of the given square, cutting the semicircumference at the points D and F; then through F draw FG perpendicular to AB, and AG and GB will be the sides of the required rectangle.

For, FG is a mean proportional between AG and GB

(B. IV, Prop. XVIII, Cor. 1); therefore $AG \times GB = FG^2$. But FG is equal to a side of the given square C; consequently $AG \times GB =$ the square C. Therefore the given line AB has been divided at the point G, so that the rectangle of the two parts is equal to the square C.

*Schol.* When the side of the given square is greater than half the given line AB, the line DF, drawn as above directed, will not cut the semicircumference; so that in this case the problem would be *impossible*.

(78.) PROBLEM. *To find the square and higher powers of a given quantity or magnitude.*



Draw the straight line AB of indefinite length, on which take the distance AC equal to a unit; through the point C, draw CD perpendicular to AB, and of indefinite length. With A as a centre, and with a radius equal to the given line, describe an arc cutting CD at the point F: through A and F, draw the line AG, of indefinite length; also draw FH perpendicular to AG, HK perpendicular to AB, KL perpendicular to AG, LM perpendicular to AB, etc.; then will AH equal the square of AF, AK will equal the cube of AF, AL will equal the fourth power of AF, etc.

For, from the manner of construction, it is obvious that the triangles ACF, AFH are right-angled and similar; hence

$$AC = 1 : AF :: AF : AH;\ \text{therefore } AH = AF^2.$$

Again, the triangles ACF and AHK are similar, and give

$$1 : AF :: AH : AK.$$

Substituting $AF^2$ for AH, we find

$$1 : AF :: AF^2 : AK; \text{ therefore } AK = AF^3.$$

Comparing the similar triangles AFC and AKL, we have

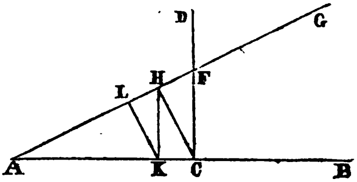
$$1 : AF :: AK : AL.$$

For AK substitute $AF^3$, and we have

$$1 : AF :: AF^3 : AL; \text{ therefore } AL = AF^4.$$

By continuing this operation, we find in succession all the different powers of AF; that is, we find all the terms of the geometrical progression 1, AF, $AF^2$, $AF^3$, $AF^4$, $AF^5$, &c.

(79.) When the given magnitude is less than a unit, we may use the following construction:



Draw the straight line AB of indefinite length; take AC equal to the given magnitude, and through C draw CD at right angles to AB, of indefinite length. Then with A as a centre, and with a radius equal to a unit, describe an arc cutting CD at the point F: through F draw the line AG, of indefinite length; also draw CH perpendicular to AG, HK perpendicular to AB, KL perpendicular to AG, etc.; then will AH equal the square of AC, AK will equal the cube of AC, AL will equal the fourth power of AC, etc.

For, comparing the triangles ACF, ACH, which are similar, we have $AF = 1 : AC :: AC : AH$; therefore $AH = AC^2$.

Again, comparing the triangles ACF, AHK, we have

$$1 : AC :: AH : AK.$$

Substituting $AC^2$ for AH, we have

$$1 : AC :: AC^2 : AK; \text{ consequently } AK = AC^3.$$

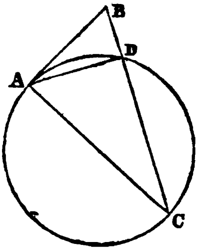
And by a similar process we can find the higher powers of AC; that is, we can find all the terms of the geometrical progression

1, AC, $AC^2$, AC,$^3$, $AC^4$, &c.

In this case, these terms form a decreasing geometrical progression; while, in the first case, the terms form an increasing geometrical progression.

### PROPOSITION XXI.

**Theorem.** *If, from the same point without a circle, a tangent and secant line be drawn, the tangent will be a mean proportional between the secant and its external segment.*



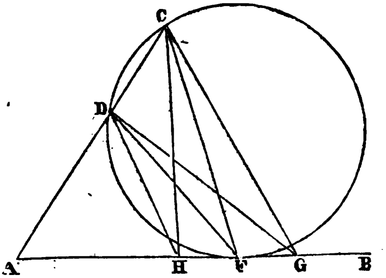
From the point B, let the tangent BA, and the secant BC, be drawn; then will BC : BA : : BA : BD,
or $BA^2 = BC.BD$.

For, drawing AD and AC, the triangles ABD and ABC will have the angle at B common; the angle ACB has for its measure half the arc AD (B. III, Prop. VIII); the angle BAD also has for its measure half the arc AD (B. III, Prop. IX); therefore the two triangles are similar, and we have

$$BC : BA :: BA : BD,$$

which immediately gives $BA^2 = BC.BD$.

(80.) **Problem.** *In a given straight line* AB, *required to find a point from which the angle subtended by another given line* CD *may be the greatest possible.*



Describe the circle CDF so as to pass through the points C and D, and touch the line AB at F; then F will be the point sought.

For, drawing CF and DF, the angle CFD will be measured by half the arc CD. But if we take the point G at the right of F, the angle CGD will be measured by half the difference of the arcs included between its sides (B. III, Prop. xiv); therefore the angle CGD is less than the angle CFD. In the same way, if we take a point H on the left of F, we can show that the angle CHD is also less than CFD. Consequently the angle CFD is the greatest angle which the line CD can subtend from any point in the straight line AB.

We will now show how the point F can be found without the aid of the circle. If CD be produced to meet AB at A, we shall have AC a secant line, and AF a tangent; consequently AF is a mean proportional between AC and AD (B. IV, Prop. xxi). Hence the distance AF may be found, when the lines AC, AD are given, by the aid of B. IV, Prop. xix.

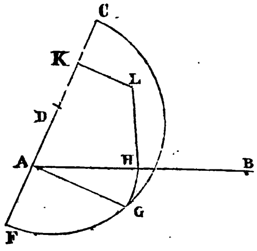
*Remark.* So long as the points A, D and C remain the same, the distance AF will be constant, whatever angle the line BA may make with the line AC.

The above problem is equivalent to the following:

*Through any two points, as* C *and* D, *to describe the circumference of a circle which shall touch a given line, as the line* AB.

From what has already been done, it is obvious we may find the centre of the required circle by the following simple method:

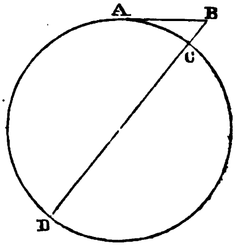
Join C and D, and produce CD beyond the line AB until the part AF is equal to AD. On CF describe the semicircumference CGF, and from A draw AG perpendicular to CF, meeting the circumference at G; then with A as a centre, and with AG as a radius, describe an arc cutting the given line AB at H. Through H, perpendicular to AB, draw HL; also through K the middle of CD, draw KL perpendicular to CD; then will L be the centre of the circle required, and LH its radius.



For, by construction, AG is a mean proportional between AC and AF, or between AC and AD; but AH is equal to AG, and therefore

AH is a mean proportional between AC and AD, which corresponds with the first solution, so that H is the point sought.

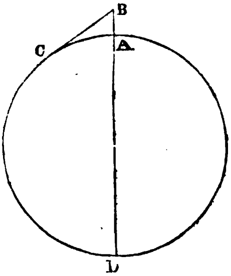
(81.) It has been found by direct measurement, upon the surface of tranquil waters, and other level extended portions of the earth, that a tangent line deviates from the curved surface of the earth by 8 inches in the distance of one mile. How may the diameter of the earth be found from this simple fact?



Suppose AB to be a tangent to the surface of the earth, one mile in length; and suppose BC to be its deviation from the surface of the earth. Then we shall have $AB^2 = BC \times BD$ [B. IV, Prop. XXI], and $BD = \frac{AB^2}{BC}$. Substituting 1 mile for AB, and 8 inches $= \frac{1}{7920}$ of a mile for BC, we find

$BD = 1 \div \frac{1}{7920} = 7920$ miles, which may be taken for the earth's diameter, since the difference BC is only 8 inches.

(82.) **Problem.** *From a given height above the surface of the earth, to find at what distance an object can be seen when in the horizon.*



Let AD the diameter of the earth be denoted by D, AB the height above the surface of the earth by $h$, and BC the distance sought by $d$. Then, since BC is a tangent and BD a secant, we have

$BC^2 = BA \times BD$; [B. IV, Prop. XXI.]

or, in symbols,

$$d^2 = h(D + h) = hD + h^2.$$

Consequently $d = \sqrt{hD + h^2} = \sqrt{hD} \times \sqrt{1 + \frac{h}{D}}$.

The fraction $\frac{h}{\mathrm{D}}$ is exceedingly small, and may, without any appreciable error, be omitted; by which means, we have

$$d = \sqrt{h\mathrm{D}}.$$

Now, if $d$ and D are taken in miles and $h$ in feet, we must have $h$ feet $= \frac{h}{5280}$ miles; so that if we consider the diameter of the earth to be 7920 miles, our expression will become

$$d = \sqrt{\frac{h}{5280} \times 7920} = \sqrt{\tfrac{3}{2}h} = \sqrt{h + \tfrac{1}{2}h}.$$

*Hence, if we increase the height (estimated in feet) by its half, and extract the square root, we shall obtain the distance in miles to which an object can be seen in the horizon.*

The above equation readily gives $h = \frac{2}{3}d^2$.

*Hence, if we know the distance to a point in the horizon, in miles, we may find our height, in feet, by taking two-thirds of the square of this distance.*

As an example, suppose the eye of an observer is 6 feet above the surface of the earth, to what distance can he see?

To 6 adding its half, we have 9; extracting the square root of 9, we find 3 for the number of miles seen.

As a second example, At what height must the eye of an observer be placed, so as to see just 10 miles?

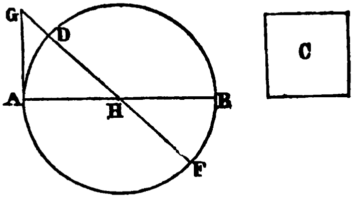
Squaring 10, we have 100; taking two-thirds of this, we find 66⅔ for the number of feet required.

(83.) It has already been remarked, that the error committed by neglecting the fraction $\frac{h}{\mathrm{D}}$ is inappreciable. But we have neglected another cause which materially affects the accuracy of our results; that is, the refraction of light, which always causes bodies to appear more elevated than they really are.

### PROPOSITION XXII.

**Problem.** *To construct a rectangle that shall be equivalent to a given square, having the difference of its adjacent sides equal to a given line.*

Let C be the given square, and AB equal to the difference of the sides of the required rectangle.



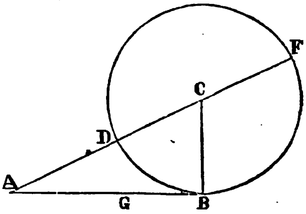
Upon AB describe the circumference ADBF; and through A draw perpendicular to AB the line AG, equal to a side of the given square C. Through the point G and the centre H draw the secant GF, cutting the circumference at D; then will GF and GD be the sides of the required rectangle.

The difference of these lines is equal to the diameter DF or AB; also $GF \times GD = AG^2$. But AG is equal to a side of the given square; therefore

$$GF \times GD = \text{the square C}.$$

### PROPOSITION XXIII.

**Problem.** *To divide a given line into two parts, such that the greater part shall be a mean proportional between the whole line and the other part.*



Let AB be the given line.

At the extremity B draw BC at right angles to AB, and equal to half the line AB; from the point C as a centre, with the radius CB, describe the circle DBF; draw AC, cutting the circumference at D and F; then take AG equal to AD. The line AB will be divided at the point G in the manner required; that is, we shall have

AB : AG : : AG : GB.

For, AB being perpendicular to the radius CB at its extremity B, is a tangent, and AF is a secant; hence we have AF : AB : : AB : AD; [B. IV, Prop. XXI.] consequently AF—AB : AB : : AB—AD : AD. Now, since the radius is half of AB, the diameter DF is equal to AB; consequently AF — AB = AD = AG; AB — AD = GB, and therefore we have

AG : AB : : GB : AG; or, exchanging the means for the extremes, we finally obtain

AB : AG : : AG : GB.

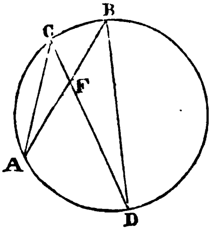
*Scholium.* This sort of division of the line AB is called *division in extreme and mean ratio.* It may be observed, that the secant AF is also divided in extreme and mean ratio; for, since AB = DF, the proportion

AF : AB : : AB : AD,

becomes AF : DF : : DF : AD.

### PROPOSITION XXIV.

THEOREM. *The segments of two chords, which intersect each other in a circle, are reciprocally proportional.*



Let the chords AB and CD intersect at F; then will

AF : DF : : FC : FB.

Draw AC and BD. In the triangles AFC, DBF, the angles at F are equal, being vertical; the angle at A is equal to the angle at D, because each is measured by half the arc BC (B. III, Prop. VIII); for a similar reason, the angle at C is equal to the angle at B. Therefore the two triangles AFC, DBF are similar, and we have

AF : DF : : FC : FB.

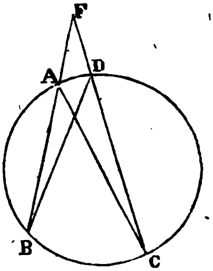
*Cor.* If we take the products of the means and extremes of the above proportion, we shall have

$$AF.FB = DF.FC;$$

that is, the rectangle of the two segments of the one chord is equal to the rectangle of the two segments of the other chord.

### PROPOSITION XXV.

THEOREM. *If, from the same point without a circle, two secants be drawn terminating in the concave arc, the whole secants will be reciprocally proportional to their external segments.*



Let the secants FB, FC be drawn from the point F; then will

$$FB : FC :: FD : FA.$$

For, drawing AC, BD, the triangles FAC, FBD have the angle at F common, and the angle at C equal to the angle at B, since each is measured by half the arc AD (B. III, Prop. VIII); therefore these triangles are similar, and we have

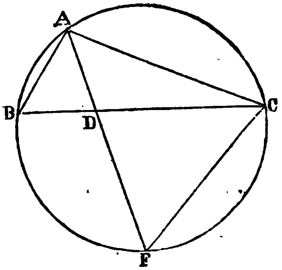
$$FB : FC :: FD : FA.$$

*Cor.* If we take the products of the means and extremes of the above proportion, we shall have

$$FB.FA = FC.FD.$$

### PROPOSITION XXVI.

THEOREM. *If either angle of a triangle is bisected by a line terminating in the opposite side, the rectangle of the sides including the bisected angle is equal to the square of the bisecting line, together with the rectangle contained by the segments of the third side.*



Let AD bisect the angle A: then will

$$AB \times AC = AD^2 + BD \times DC.$$

Describe a circumference through the three points B, A, C; produce AD till it meets this circumference at F, and join CF.

The triangle BAD is similar to the triangle FAC; for, by hypothesis, the angle BAD = FAC; also the angle B = F, each being measured by half the arc AC. Hence these triangles are similar, and we have

$$BA : AF :: AD : AC; \text{ which gives}$$
$$BA \times AC = AF \times AD;$$

or, using AD + DF for AF, we have

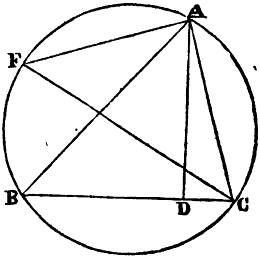
$$BA \times AC = AD^2 + AD \times DF.$$

But AD × DF = BD × DC [B. IV, Prop. XXIV, Cor.]; therefore we finally obtain

$$BA \times AC = AD^2 + BD \times DC.$$

### PROPOSITION XXVII.

**Theorem.** *In every triangle, the rectangle contained by any two sides is equal to the rectangle contained by the diameter of the circumscribing circle, and the perpendicular drawn to the third side from the opposite angle.*



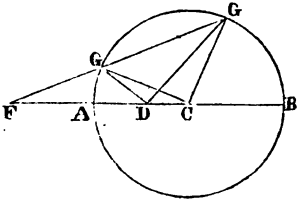
In the triangle ABC, let AD be drawn perpendicular to BC, and let CF be the diameter of the circumscribed circle; then will

$$AB \times AC = AD \times CF.$$

For, joining AF, the triangles ABD, AFC are right-angled, the one at D, the other at A; also the angle B = F, each being measured by half the arc AC (B. III, Prop. VIII). Hence the triangles are similar, and we have AB : CF :: AD : AC; consequently, $AB \times AC = AD \times CF$.

## PROPOSITION XXVIII.

THEOREM. *If a point be taken on the radius of a circle, and this radius be then produced, and a second point be taken on it without the circumference, these points being so situated that the radius of the circle shall be a mean proportional between their distances from the centre, then, if lines be drawn from these points to any points of the circumference, the ratio of such lines will be constant.*



Let D be the point within the circumference, and F the point without; then, if CD : CA : : CA : CF, the ratio of FG to GD will be the same for all positions of the point G.

For, by hypothesis, CD : CA : : CA : CF, or substituting CG for CA, CD : CG : : CG : CF; hence the triangles CDG and CFG have each an equal angle C contained by proportional sides, and are therefore similar (B. IV, Prop. x), and the third side GD is to the third side GF as CD to CG or CA. But, by division, the proportion

CD : CA : : CA : CF gives
CD : CA : : CA − CD : CF − CA, or
CD : CA : : AD : AF; therefore,
GD : GF : : AD : AF; and since the ratio of AD to AF is constant, it follows that the ratio of GD to GF is also constant.

# BOOK FIFTH.

## DEFINITIONS.

1. Any polygonal figure is said to be *equilateral*, when all its sides are equal; and it is *equiangular*, when all its angles are equal.

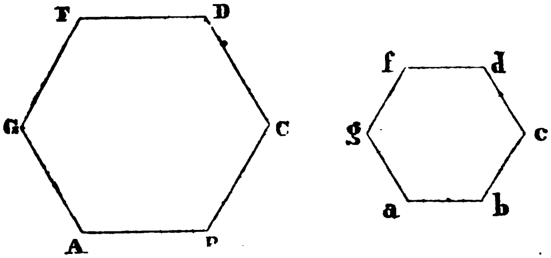
2. Two polygons are said to be *mutually equilateral*, when their corresponding sides, taken in the same order, are equal. When this is the case with the corresponding angles, the polygons are said to be *mutually equiangular*.

3. A *regular polygon* has all its sides and all its angles equal. If all the sides and all the angles are not equal, the polygon is *irregular*.

4. A regular polygon may have any number of sides not less than three. The *equilateral triangle* (Def. XV) is a regular polygon of three sides. The *square* (Def. XIX) is also a regular polygon of four sides.

### PROPOSITION I.

**Theorem.** *Two regular polygons of the same number of sides, are similar figures.*



Suppose we have, for example, the two regular hexagons ABCDFG, *abcdfg ;* then will these two polygons be similar figures.

For, the sum of all the angles is the same in the one as the other (B. I, Prop. XXIV). In this case the sum of all the angles is eight right angles; the angles are therefore each equal to one-sixth of eight right angles, and hence the two polygons are equiangular.

Again, since AB, BC, CD, etc. are equal, and *ab*, *bc*, *cd*, etc. are also equal, we have

AB : *ab* : : BC : *bc* : : CD : *cd*, etc. It therefore follows that two regular polygons of the same number of sides have equal angles, and the sides about those equal angles proportional; consequently they are similar (B. IV, Def. 3).

*Cor.* The perimeters of two regular polygons having the same number of sides, are to each other as their homologous sides; and their areas are to each other as the squares of those sides (B. IV, Prop. XVI).

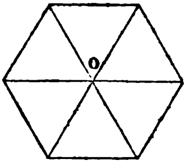
(84.) It has been shown (B. I, Prop. XXIV), that the sum of all the interior angles of a polygon is found by multiplying two right angles, or 180°, by a number which is two less than the number of sides of the polygon; or, if $n$ denote the number of sides, then the sum of all the angles will be $(n-2)180°$. But, since all the angles of a regular polygon are equal, the magnitude of one of these angles may be found by dividing the sum of all by the number of angles, or, which is the same, by the number of sides in the polygon; therefore each angle is equal to $\frac{n-2}{n}.180°$, or $180° - \frac{360°}{n}$.

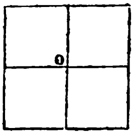
From this formula, the angles of regular polygons, from the equilateral triangle upwards, have been calculated as in the following table:

| No. of sides. | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| Mag. of angle. | 60° | 90° | 108° | 120° | $128\frac{4}{7}°$ | 135° | 140° | 144° | $147\frac{3}{11}°$ | 150° |

(85.) The expression for the angle of a regular polygon, which is $180° - \frac{360°}{n}$, shows that no polygon can have angles consisting of a whole number of degrees, unless the number of sides is an exact divisor of 360. Now the divisors of 360 are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360 (see *Higher Arithmetic*). Excluding 1 and 2, which evidently cannot give polygons, since a polygon cannot have less than 3 sides, we have twenty-two divisors left; therefore there are only twenty-two regular polygons, whose angles are expressed by a whole number of degrees. Had the French method of dividing the circumference into 400 degrees been adopted, the number of regular polygons, having an integral number of degrees in each angle, would have been only thirteen.

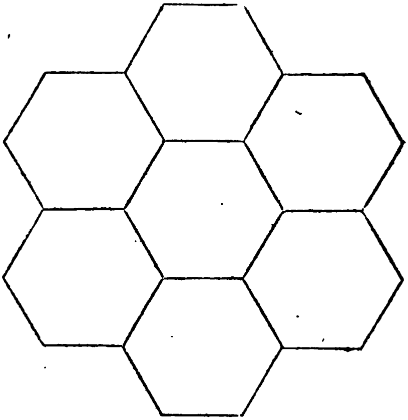
(86.) In ornamental architecture, polygons are used in the formation of surfaces produced by the juxtaposition of solid blocks, as in flooring, paving, or by their superposition as in masonry. The polygons used in such cases must always be such as will admit of being put together without leaving open spaces between them. If they be laid together, as is sometimes the case, leaving the vertices of their angles coincident, then no regular figures can be used, except those whose angles are of such a magnitude as will exactly fill the space surrounding a point. It is evident that the equilateral triangle and square will fulfil this condition, since six angles of an equilateral triangle, and four of a square, make up exactly 360°; thus the point O, in the first figure, is surrounded by six equilateral triangles, and in the second figure it is surrounded by four squares.



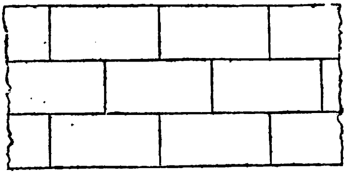


In general, the condition necessary to be fulfilled is that......... $180^\circ - \frac{360^\circ}{n}$ should be a divisor of 360° without a remainder; or, dividing both terms of this expression by 180°, the condition will be that $1 - \frac{2}{n}$ shall be a divisor of 2. The only whole numbers for $n$ which will fulfil this condition, are 3, 4 and 6; hence it follows that a surface cannot be completely covered by any regular figures except the equilateral triangle, the square, and the hexagon.

The angles of the hexagon being 120°, three of them will fill the space round a point, as here represented.



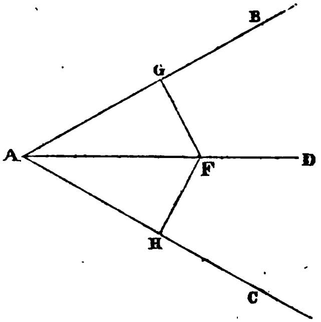
In the formation of pavement, it is an object to avoid the combination of a great number of angles at the same point; the strength of the surface being weakened thereby, and the liability to fracture increased. The combination of equilateral triangles is objectionable on these grounds; and even the combination of squares is usually avoided, by causing the angles at which each pair of adjacent sides are united to coincide with the middle of the sides of a succeeding series, as here represented.



Where the angles of the component figures are intended to be invariably combined, the hexagonal arrangement will therefore have greater strength and stability for pavement than the others; but for upright masonry, the square or rectangular division is preferable, since the surfaces of contact take the position best adapted to sustain the incumbent weight of the structure.

### PROPOSITION II.

**Theorem.** *If a line be drawn bisecting an angle formed by two given lines, every point in this bisecting line will be equally distant from the two lines forming the angle.*



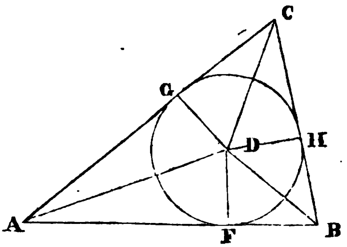
Let AB, AC form the angle BAC, which is bisected by the line AD; then will every point in AD be equally distant from AB, AC.

For, in the line AD take any point as F, and draw FG, FH perpendicular respectively to AB, AC. Then, comparing the two triangles AFG, AFH, we see that the angles AGF and AHF are equal, each being a right angle; also the angles FAG, FAH are equal, since AF bisects the angle GAH : hence the angle AFG is equal to AFH (B. I, Prop. xxiv, Cor. 1). Therefore in these two triangles we have the side AF common, and the two adjacent angles equal; consequently they are identical (B. I, Prop. iv), and FG is equal to FH.

*Cor.* The centres of all the different circles which can be described, touching the two lines AB, AC, must be situated in the line AD, which bisects the angle BAC.

### PROPOSITION III.

**Problem.** *To inscribe a circle in a given triangle.*



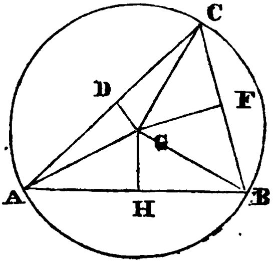
Let ABC be the given triangle.

Bisect the angle BAC by the line AD; also bisect the angle ABC by the line BD (B. I, Prop. xi). Then, if from the point D where these bisecting lines intersect, as a centre, a circumference be described touching AB, it will also touch AC and BC (B. V, Prop. ii, Cor.), and consequently be the inscribed circle required.

*Cor.* The three lines bisecting the three angles of a triangle meet at the same point, which point is the centre of the inscribed circle.

### PROPOSITION IV.

**Problem.** *To circumscribe a circle about a given triangle.*



Let ABC be the given triangle.

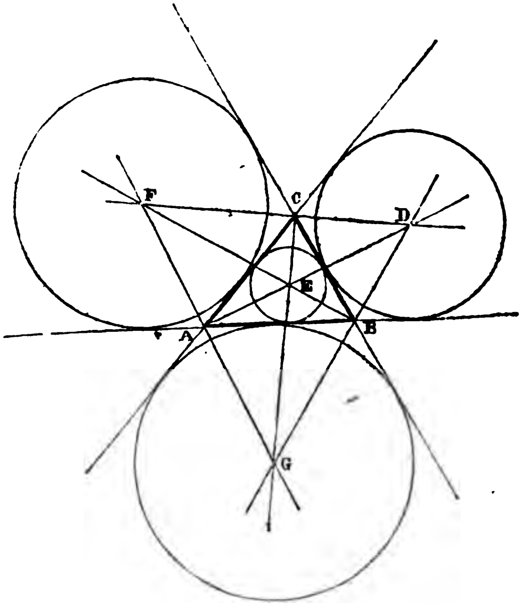
Bisect the sides CA, CB with the perpendiculars DG, FG, which must meet in some point G; for, if they did not meet, they would be parallel, and, if parallel, then also would the lines to which they are perpendicular be parallel; that is, CA, CB would be parallel, which is not the case: consequently the perpendiculars DG, FG must meet. Join GA, GB, GC; and comparing the two right-angled triangles DGC, DGA, we have DG common, and the side DC equal to DA, each being half the side AC; therefore GA is equal to GC (B. II, Prop. VIII, Cor. 2). By comparing the two triangles HGA, HGB, we may, for similar reasons, show that GA is also equal to GB. Therefore the point G is equidistant from A, B and C.

Hence, if, with G as a centre, a circumference be described passing through A, it will also pass through B and C, and consequently circumscribe the triangle ABC.

*Cor.* 1. The three perpendiculars which bisect the three sides of a triangle, meet in the same point.

*Cor.* 2. This problem is equivalent to describing a circumference of a circle through any three points not situated in the same straight line.

(87.) A circle described so as to touch one of the sides of a triangle exteriorly, and the other two sides produced, is called an ***escribed circle;*** thus, D, F and G are centres of escribed circles, in reference to the triangle ABC.



From this we see that escribed circles touch the three sides of a triangle, and are situated wholly without the triangle; while the inscribed circle also touches all the sides of a triangle, but is situated wholly within the triangle.

(88.) Hence, four circumferences may always be described, which shall touch any three given lines, provided no two of the lines are parallel or coincident; for these lines, being sufficiently produced, will form, by their intersections, a triangle; which triangle we have just shown has three escribed circles and one inscribed circle, making in all four circles, each of which touches the three given lines.

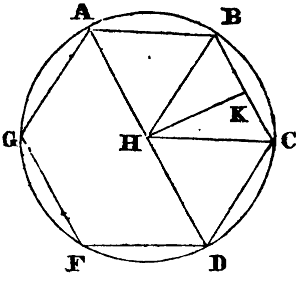
(89.) We have shown that the three lines which bisect the three angles of a triangle, meet at the centre of the inscribed circle (B. V, Prop. III, Cor.). It also follows (B. V, Prop. II, Cor.), that these lines, when produced, pass through the centres of the escribed circles. It is moreover obvious, that if the exterior angles of a triangle be bisected, these bisecting lines will meet at the centres of the escribed circles. Hence, if lines be drawn bisecting the angles, and the exterior angles of a triangle, they will intersect each other by threes at the centres of the inscribed and escribed circles; thus the three lines which bisect the angles will meet at the same point, giving the centre of the inscribed circle. Any one of the lines bisecting an angle of the triangle will intersect, at the same point, two of the lines which bisect the exterior angles, giving the centre of an escribed circle.

### PROPOSITION V.

THEOREM. *A circle may be circumscribed about any given regular polygon; so also may a circle be inscribed in any given regular polygon.*

Let ABCDFG be a regular polygon.

Through any three consecutive corners of the polygon, as A, B, C, describe the circumference of a circle (B. V, Prop. IV), and it will also pass through all the remaining corners of the polygon.



For, from the centre H, draw HA, HB, HC, HD; also draw HK perpendicular to the side BC. Now, comparing the two quadrilaterals HKBA, HKCD, we know that the angle HKB is equal to HKC, since each is a right angle;

also the angle KBA is equal to KCD, since the polygon is regular : moreover KB is equal to KC (B. III, Prop. I). Therefore, if the quadrilateral HKBA be applied to the quadrilateral HKCD, so that HK may retain its present position, the side KB will coincide with KC, and BA will take the direction of CD ; and since BA is equal to CD, the point A will coincide with D : consequently the point H is equidistant from A, B, C and D. In the same way all the corners of the polygon may be shown to be at the same distance from H ; therefore this circle circumscribes the polygon.

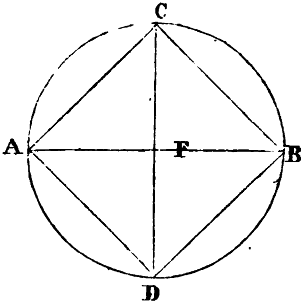
Again, in reference to this circle, the sides of the given polygon are chords; they are therefore equally distant from the centre (B. III, Prop. IV). Hence, if, with H as a centre, and with HK as a radius, a circle be described, it will touch all the sides of this polygon at their middle points, and will therefore be inscribed in it.

*Schol.* 1. The point H, which is the common centre of the circumscribed and inscribed circles of the polygon, may be regarded as the centre of the polygon itself; and, on this principle, the angle AHB is called *the angle at the centre*, being formed by the two radii drawn to the extremities of the same side. And since all the chords AB, BC, CD, &c. are equal, all the angles at the centre must be equal ; and therefore each may be found by dividing four right angles by the number of sides of the polygon.

*Schol.* 2. In order to inscribe a regular polygon of a certain number of sides, in a given circle, we must be able to divide the circumference into as many equal parts as the polygon is to have sides ; and then, by joining these successive points of division, we shall form the polygon desired. For the arcs being equal, their chords must be equal; also the angles of the polygon will be equal, since they will be inscribed in equal segments of the same circle (B. III, Prop. VIII, Cor. 1).

### PROPOSITION VI.

PROBLEM. *To inscribe a square in a given circle.*



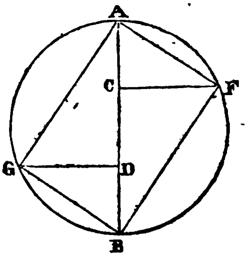
Draw the two diameters AB, CD, cutting each other at right angles; join the extremities, and the figure ACBD will be a square. For the angles ACB, CBD, BDA & DAC are each right angles, being inscribed in a semicircle (B. III, Prop. VIII, Cor. 3); hence the figure is equiangular. Again, since the arcs AC, CB, BD, DA are quadrants, they are equal, and therefore their corresponding chords are equal; so that the figure is also equilateral, and consequently it is a square.

*Scholium.* Since the triangle AFC is right-angled and isosceles, we have (B. II, Prop. VIII) $AC^2=2\,AF^2$. Hence

$$AC^2 : AF^2 :: 2 : 1, \text{ and}$$

$$AC : AF :: \sqrt{2} : 1;$$ that is, the side of an inscribed square is to the radius, as the square root of 2 is to 1.

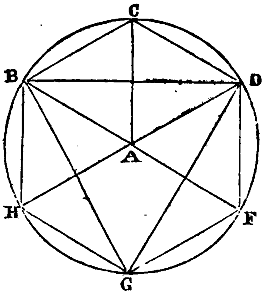
(90.) The strongest rectangular beam that can be sawed from a cylindric log, may be found by the following construction:



Let AFBG be a circular section of the log; trisect the diameter AB at the points C and D; draw CF, DG perpendicular to this diameter, and join AF, FB, BG, GA; and then will the rectangle AFBG be a section of the beam of the maximum strength. The proof of this cannot be made apparent by the elements of geometry alone.

### PROPOSITION VII.

PROBLEM. *In a given circle, to inscribe a regular hexagon, and an equilateral triangle.*



Draw the radius AB; make the chord BC equal to this radius, and join AC. Then, since the triangle ABC is equilateral, each of its angles is equal to $\frac{1}{3}$ of two right angles, or $\frac{1}{6}$ of four right angles. Now since all the angular space about the point A is equal to four right angles (B. I, Prop. I, Cor. 3), it follows that the arc BC is one-sixth of the entire circumference. Hence if we apply the radius of the circle six times upon the circumference, it will bring us round to the place of departure, and we shall thus inscribe the hexagon as required.

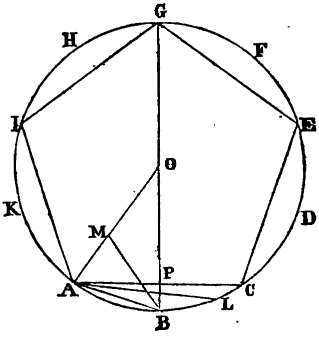
If we join the alternate angles of the hexagon, we shall form the inscribed equilateral triangle BDG.

*Scholium.* The figure ABCD is a parallelogram having equal sides, since each side is equal to the radius; hence we have $BD^2+AC^2 = 4\,AB^2$. [B. II, Prop. xiv.] Rejecting $AC^2$ from the first member of this equation, and its equal $AB^2$ from the second member, we have $BD^2 = 3\,AB^2$.
Hence $BD^2 : AB^2 :: 3 : 1$, and
$BD : AB :: \sqrt{3} : 1$; that is, the side of the inscribed equilateral triangle is to the radius, as the square root of 3 is to 1.

Comparing this scholium with the scholium of the last proposition, we see that the side of an inscribed square is to the side of an inscribed equilateral triangle, as the square root of 2 to the square root of 3.

### PROPOSITION VIII.

PROBLEM. *To inscribe in a given circle a regular decagon, a regular pentagon, and also a regular polygon of fifteen sides.*



Divide the radius AO in extreme and mean ratio (B. IV, Prop. xxiii) at the point M; take the chord AB equal to OM the greater segment; then will AB be the side of the regular decagon, and, being applied ten times to the circumference, will give the decagon required.

For, joining MB, we have by construction

$$AO : OM :: OM : AM;$$

or, since AB = OM, we have

$$AO : AB :: AB : AM.$$

Now since the triangles ABO, AMB have a common angle A included between proportional sides, they are similar (B. IV, Prop. x); and as the triangle OAB is isosceles, it follows that AMB must also be isosceles, and AB=BM; but AB=OM, and hence MB=OM, so that the triangle BMO is also isosceles.

Again, the angle AMB being exterior to the isosceles triangle BMO, is double the interior angle O (B. I, Prop. xxv); but the angle AMB = MAB : hence the triangle OAB is such that each of the angles at its base, OAB or OBA, is double of O the angle at its vertex. Consequently the three angles of this triangle are together equal to five times the angle O, which is therefore one-fifth of two right angles, or the tenth part of four right angles; hence the arc AB is the tenth part of the circumference, and the chord AB is the side of the regular decagon.

By joining the alternate corners of the regular decagon, the pentagon ACEGI will be formed, also regular.

AB being still the side of the decagon, let AL be the side of a regular inscribed hexagon. The arc BL will, with reference to the whole circumference, be $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$; hence the chord BL will be the side of a regular polygon of fifteen sides.

*Scholium.* Any regular polygon being inscribed, if the arcs subtended by its sides be severally bisected, the

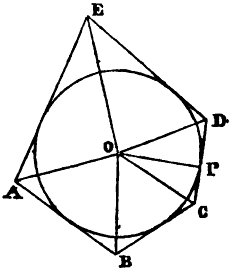
**chords of these semi-arcs will form a new regular polygon of double the number of sides.**

(91.) From the square will arise regular polygons of 8, 16, 32, &c. sides, all of which may be included in the formula $4.2^n$, when $n$ is any positive integer, including also the case of $n=0$. In like manner the equilateral triangle will give polygons expressed by $3.2^n$, when $n$ is subject to the same restrictions as in the square. Those arising from the pentagon will be included in the formula $5.2^n$; and those from the polygon of fifteen sides, will be given by $15.2^n$.

(92.) It is obvious that any regular polygon whatever might be inscribed in a circle, provided that its circumference could be divided into any proposed number of equal parts; but such division of the circumference, like the trisection of an angle, which, indeed, depends on it, is a problem which has not yet been effected. There are no means of inscribing in a circle a regular heptagon; or, which is the same thing, the circumference of a circle cannot be divided into seven equal parts, by any method hitherto discovered. Indeed the polygons above noticed were, till about the beginning of the present century, supposed to include all that could admit of inscription in a circle; but in 1801 a work was published by M. Gauss of Gottingen (and afterwards translated into French by M. Delisle, under the name of *Recherches Arithmetiques*), containing the curious discovery that the circumference of a circle could be divided into any number of equal parts capable of being expressed by the formula $2^n+1$, provided it be a prime number, that is, a number that cannot be resolved into factors. The number 3 is the simplest of this kind, it being the value of the above formula when $n=1$; the next prime number is 5, and this also is contained in the formula. But polygons of 3 and 5 sides have already been described. The next prime number expressed by the formula is 17; so that it is possible to inscribe a seventeen-sided polygon in a circle. The investigation of Gauss's theorem, although it establishes the above geometrical fact, depends upon the theory of algebraical equations, and involves other considerations of a nature that do not enter into the elements of geometry; we must, therefore, content ourselves with merely alluding to it.

### PROPOSITION IX.

THEOREM. *The surface of every polygon in which a circle may be inscribed, is equivalent to the rectangle of half the radius of that circle, and the perimeter of the polygon.*



Let O be the centre of the circle inscribed in the polygon ABCDE. Draw from O lines to the extremities of the sides, thus dividing the polygon into as many triangles as it has sides.

Then, the common altitude of these triangles is the radius OP of the circle. Hence the surface of any one of them, OCD for instance, is equivalent to the rectangle of half OP and CD (B. II, Prop. III), and so of any other; therefore the sum of all the triangles, that is, the surface of the polygon, is equivalent to the rectangle of half the radius and the whole perimeter of the polygon.

*Cor.* It has been shown (B. V, Prop. III), that a circle may be inscribed in a triangle; consequently a triangle is equivalent to the rectangle of half the radius of the inscribed circle, and its perimeter.

*Schol.* The above proposition is obviously true for all regular polygons (B. V, Prop. V).

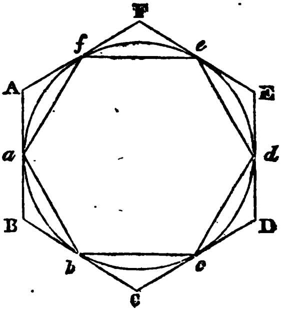
The converse of this proposition is as follows: If the surface of a polygon be equivalent to the rectangle of its perimeter and another line, this line will be half the radius of the inscribed circle, which it is obvious has not place; for the surface of any polygon, whether it can circumscribe a circle or not, may always be represented by an equivalent rectangle, of which one side may be of any given length; so that when the perimeter is one side, the other side would be half the radius of the inscribed circle, whether the polygon admit of such inscription or not. But if the converse be enunciated thus: If the surface of a polygon in which a circle may be inscribed be equivalent to a rectangle, of which one side is the perimeter, then the other side will be half the radius of the inscribed circle; or if one side be half the radius of the inscribed circle, the other will be the perimeter—its truth immediately follows from the proposition itself.

### PROPOSITION X.

PROBLEM. *An inscribed regular polygon being given, to circumscribe a similar polygon about the circle; and, conversely, from having a circumscribed regular polygon, to form the similar inscribed one.*

Let *abcd*, &c. be a regular inscribed polygon: it is required to describe a similar polygon about the circle.

At each of the points *a*, *b*, *c*, *d*, etc., draw tangents to the circle, and they will form the polygon ABCD, &c. similar to the polygon, *abcd*, &c.

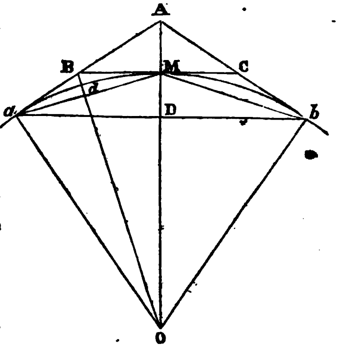


For, in the first place, there are as many tangents as the inscribed polygon has sides; and those drawn through the extremities of the same chord evidently meet, unless the chord is a diameter.

Next, the angles formed by these tangents and chords are all equal to each other, for their sides include equal arcs (B. III, Prop. VII). Hence the triangles *f*A*a*, *a*B*b*, *b*C*c*, &c. are isosceles, and they have equal bases *fa*, *ab*, *bc*, &c.; therefore these triangles are equal, and consequently the angles A, B, C, D, &c. are equal, and so are their including sides: therefore the polygon ABCD, &c. is regular, and it has the same number of sides as the polygon *abcd*, &c.; it is, therefore, similar to it (B. V, Prop. I).

*Conversely.* Let the circumscribed polygon ABCD, &c. be given; then, if the successive points of contact *a*, *b*, *c*, *d*, &c. be joined, a similar polygon will be inscribed in the circle. For the angles A, B, C, &c. are equal; as also the sides *a*B, B*b*, *b*C, C*c*, &c., each being half a side of the polygon. Consequently the sides *ab*, *bc*, *cd*, &c. are equal, and the angles included between them must be equal; they therefore form a regular polygon, and as the sides are the same in number as those of the circumscribed polygon, it is similar to it.

*Schol.* 1. It was remarked (B. V, Prop. VIII, Schol.), that, from having an inscribed regular polygon, we might easily form another of double the number of sides. It may be, in like manner, here observed, that, from having a circumscribed regular polygon, we may readily derive another of double the number of sides, nothing more being necessary than to draw tangents to the points of bisection of the arcs intercepted by the sides of the proposed polygon, limiting these tangents by those sides; and it is plain that each of the polygons so formed will be less in surface than the preceding, being entirely comprehended within it.



Let $ab$ be the side of an inscribed polygon; and if $aA$, $bA$ be tangents to the circle at the points $a$, $b$, each will be one half of the side of the similar circumscribed polygon; or, which is the same thing, they will together be equal to the side of a circumscribed polygon, similar to the inscribed one whose side is $ab$. Let M be the middle of the intercepted arc, and draw $Ma$, $Mb$, and the tangent BMC; then $aM$, $Mb$ will be two consecutive sides of an inscribed polygon, having double the number of sides that the polygon has whose side is $ab$; and, consequently, BC being a tangent at M, meeting the tangents at $a$ and $b$, must, by the proposition, be the side of a polygon having double the number of sides that the polygon has whose side is $ab$.

*Schol.* 2. If polygons be thus successively circumscribed about the circle, their perimeters will decrease as the number of sides increase. For BC is less than $AB+AC$, and consequently $aB + BC + Cb < aA + Ab$. Now $aB+Cb = BC$, and $aA+Ab$ is equal to a side of the first circumscribed polygon : hence two sides of the second circumscribed polygon are together less than one side of the first; and, therefore, the whole perimeter of the second is less than that of the first. It is obvious, that with respect to the inscribed polygons, the perimeters increase in the same circumstance; thus, the two sides $a$M, M$b$ being together longer than $ab$, it follows that the perimeter of the second inscribed polygon exceeds that of the first.

The successive circumscribed polygons that we have been considering, continually approach nearer and nearer towards coincidence with the circle. For OB is nearer an equality to the radius O$a$ of the circle, than OA, because in the two right-angled triangles O$a$B, O$a$A, each having the common side O$a$, we have $a$A longer than $a$B, and therefore OB is less than OA; and in every succeeding polygon the difference between the radius of the circle and the distance of the centre from the remotest points in the perimeter will, in like manner, perpetually diminish; so that the perimeters continually approach towards coincidence with the circumference, and we have already seen that the perimeters continually *diminish.*

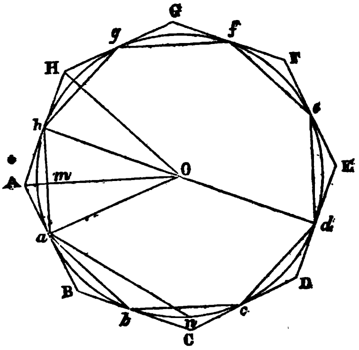
Now it is plain that if a series of magnitudes continually approach nearer and nearer towards coincidence with any proposed magnitude, and at the same time continually diminish, the magnitude to which they approach must be smaller than either of the approaching terms; we are, therefore, warranted in asserting that *the circumference of a circle is a shorter line than the perimeter of any circumscribed polygon.*

In a similar manner, by considering the successive inscribed polygons, it appears that *they* also continually approach towards coincidence with the circle. For O*d* is nearer an equality to the radius than OD, since the chord *a*M is shorter than *ab* (B. III, Prop. IV); so that in each succeeding polygon the perimeter approaches nearer to coincidence with the circumference, and it has been shown that these perimeters successively *increase*. Hence we may infer that *the circumference of a circle is a longer line than the perimeter of any inscribed polygon.*

### PROPOSITION XI.

THEOREM. *Two polygons may be formed, the one within, and the other about a circle, that shall differ from each other by less than any assigned magnitude however small.*

Let M represent any assigned surface. It is to be shown that two polygons may be described, the one within, and the other about the circle whose centre is O, which will differ from each other by a magnitude less than M.



Let N be the side of a square, whose surface is less than the surface M, and inscribe in the circle a chord *an* equal to the line N. Then, by the methods already explained, inscribe in the circle a square, a hexagon, or indeed any regular polygon : let the arcs which its sides subtend be bisected; the chords of the half arcs will be the sides of a regular polygon, having double the number of sides. Let, now, the arcs subtended by the sides of this second polygon be in like manner bisected; the chords will form a third polygon, having double the number of sides that the second has. Continue these successive bisections till the arcs become so small as to be each less than the arc *an*, their chords forming the inscribed polygon *abcd*, &c. Circumscribe the circle with a similar polygon ABCD, &c.; then this last will exceed the former by a magnitude less than the proposed magnitude M.

From the centre O draw the lines O*a*, OA, O*h*, OH, and produce *h*O to *d*; then the polygon *abcd*, &c. is composed of as many triangles equal to O*ah* as the polygon has sides; and in like manner the polygon ABCD, &c. is composed of as many triangles equal to OAH as this polygon has sides; and as the polygons have each the same number of sides, the inscribed is the same multiple of the triangle O*ah* that the circumscribed is of the triangle OAH. Now the triangle O*mh* is half the triangle O*ah*; in like manner, the triangle O*h*A is half the triangle OAH : hence the inscribed polygon is the same multiple of O*mh*, that the circumscribed polygon is of O*h*A; and, consequently,

$$Omh : OhA :: \text{ins. pol.} : \text{circ. pol.}$$

Whence, by division, we have

$$OhA : OhA - Omh :: \text{circ. pol.} : \text{circ. pol.} - \text{ins. pol.};$$

that is, $OhA : Amh :: \text{circ. pol.} : \text{circ. pol.} - \text{ins. pol.}$

Now $OhA$ is a right angle; and since AO bisects the angle A of the isosceles triangle $aAh$, it is perpendicular to $ah$: therefore the triangles $OhA$, $Amh$ are similar. Consequently (B. IV, Prop. XII),

$$OhA : Amh :: Oh^2 : hm^2 :: hd^2 : ha^2;$$ whence

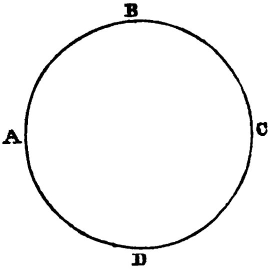
$$hd^2 : ha^2 :: \text{circ. pol.} : \text{circ. pol.} - \text{ins. pol.}$$

Now a circumscribed square, that is to say $hd^2$, is greater than the polygon ABCD, &c., since the surfaces of circumscribed polygons diminish as their sides increase in number; so that in the last proportion, the first antecedent is greater than the second: consequently the first consequent is greater than the second; that is, the excess of the circumscribed polygon above the inscribed is less than $ha^2$, and therefore less than $N^2$ or than M.

*Cor.* As the circle is obviously greater than any inscribed polygon and less than any circumscribed one, it follows that *a polygon may be inscribed or circumscribed, which will differ from the circle by less than any assignable magnitude.*

### PROPOSITION XII.

**THEOREM.** *A circle is equivalent to the rectangle contained by lines equal to the radius and half the circumference.*



Let us represent the rectangle of the radius and semicircumference of the circle ABCD by P : we are to show that this rectangle is equal in surface to the circle.

If the rectangle P be not equivalent to the circle, it must be either greater or less. Suppose it to be greater, and let us represent the excess by Q. Then, by the corollary to last proposition, a polygon may be circumscribed about the circle, which shall differ therefrom by a magnitude less than Q, and must consequently be less than the rectangle P. But the area of every circumscribed polygon is equivalent to the rectangle of the radius and half its perimeter (B. V, Prop. IX), and the perimeter exceeds the circumference of the circle : consequently the rectangle of the radius of the circle and semi-perimeter of the polygon must be greater than P, the rectangle of the same radius and semicircumference of the circle ; but it was shown above to be less, which is absurd. Hence the hypothesis that P is greater than the circle, is false.

But suppose the rectangle P is less than the circle, and let us represent the defect by the same letter Q. Then, by the same corollary, a regular polygon may be inscribed in the circle, which shall differ from it by a magnitude less than Q, and must consequently be greater than the rectangle P. But every inscribed regular polygon is equivalent to the rectangle of the perpendicular drawn from the centre to one of the sides, into half its perimeter; and this perpendicular is less than the radius of the circle, and the perimeter is less than the circumference : consequently the rectangle of this perpendicular and semi-perimeter of the polygon must be less than P, the rectangle of the radius and semicircumference of the circle; but it was shown above to be greater, which is absurd. Hence the second hypothesis also is false.

As therefore the circle can be neither greater nor less than the rectangle P, it must necessarily be equivalent to it.

### PROPOSITION XIII.

THEOREM. *Circles are to each other as the squares of their radii.*

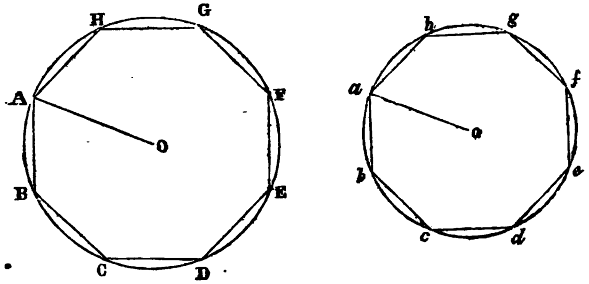
Let the circles ABCD, *abcd*, be compared : we shall have the proportion

$$AO^2 : ao^2 :: \text{circle } ABCD : \text{circle } abcd.$$

For if this proportion has not place, let there be

$$AO^2 : ao^2 :: \text{circle } ABCD : P,$$

P being some magnitude either greater or less than the circle *abcd*. Suppose it to be less, and let us represent the defect by Q. Then (B. V, Prop. XI) a polygon may be inscribed in the circle *abcd*, which shall differ from it by a magnitude less than Q, and will therefore exceed the magnitude P. Let *abcde*, &c. be such a polygon; and describe a similar polygon ABCDE, &c. in the other circle. Then, since regular polygons of the same number of sides are similar, we have



$$AO^2 : ao^2 :: \text{pol. ABCDE, \&c.} : \text{pol. } abcde\text{, \&c.}$$

Hence, by equality of ratios, we have

$$\text{circ. ABCD} : P :: \text{pol. ABCDE, \&c.} : \text{pol. } abcde\text{, \&c.}$$

Now in this proportion the first antecedent is greater than the second, consequently the first consequent is greater than the second; that is, P is greater than the polygon *abcde*, &c.; but it has been shown to be less, which is absurd. Therefore P cannot be less than the circle *abcd*.

But suppose that P is greater than the circle *abcd*. Then, still representing the difference by Q, a polygon may be circumscribed about the circle *abcd*, which shall differ from it by a magnitude less than Q, or be less than P. Suppose such a polygon to be described, and that a similar one is formed about the circle ABCD; then these polygons being to each other as the squares of the radii of their respective circles, it will evidently result, by comparing, as in the preceding case, this proportion with that advanced in the hypothesis, that the circle ABCD is to P as the polygon about this circle to the polygon about the other; in which proportion the first antecedent is less than the second, and consequently the first consequent is less than the second, that is, P is less than the polygon circumscribed about the circle *abcd*; but it was shown above to be greater, which is absurd. Hence P can neither be less nor greater than the circle *abcd*; consequently it must be equal to it, and therefore

$$AO^2 : ao^2 :: \text{circle } ABCD : \text{circle } abcd.$$

*Cor.* 1. Since every circle is equivalent to the rectangle of its radius and half its circumference, the above proportion may be expressed thus:

$$AO^2 : ao^2 :: AO.\tfrac{1}{2}ABCD : ao.\tfrac{1}{2}abcd;$$

whence, $AO : ao :: \tfrac{1}{2}ABCD : \tfrac{1}{2}abcd.$

Consequently *the circumferences of circles are to each other as their radii*, and therefore *their surfaces are as the squares of the circumferences.*

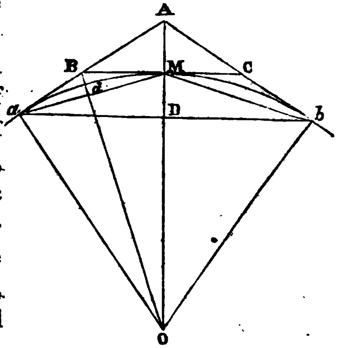
*Cor.* 2. It follows, also, that *similar arcs are to each other as the radii of the circles to which they belong*; for they subtend equal angles at the centres (B. III, Def. 12), and each angle is to four right angles as the arc which subtends it is to the whole circumference: consequently the one arc is to the whole circumference, of which it forms a part, as the other arc to the circumference of which it is part; and as the circumferences are as the radii, we have alternately the one arc to the other as the radius of the former to that of the latter.

*Cor*. 3. Therefore also *similar sectors are to each other as the squares of their radii;* for each sector is to the circle as the arc to the circumference : consequently the one sector is to its circle as the other sector to its circle ; and as the circles are as the squares of the radii, we have alternately the one sector to the other as the square of the radius of the former to the square of that of the latter.

*Cor*. 4. It readily follows that *similar segments are also as the squares of the radii;* for they result from similar sectors, by taking away from each the triangle formed by the chord and radii, which triangles, being similar, are also to each other as the squares of the radii; therefore the sectors and triangles being proportional, it follows that the segments also are as the sectors, or as the squares of the radii, or indeed as the squares of their chords.

### PROPOSITION XIV.

PROBLEM. *The surface of a regular inscribed polygon and that of a similar circumscribed polygon being given, to find the surfaces of regular inscribed and circumscribed polygons of double the number of sides.*



Let $ab$ be a side of the given inscribed polygon: then the tangents $a$A, $b$A will each be half the side of the similar circumscribed polygon; the chords $a$M, $b$M, to the middle of the arc $a$M$b$, will be sides of an inscribed polygon of double the number of sides; and, lastly, the tangent BMC will be the side of a circumscribed polygon similar to this last. All this is evident from B. V, Prop. x.

Let us, now, in order to avoid confusion, denote the inscribed polygon whose side is $ab$ by $p$, the corresponding circumscribed polygon by P; the inscribed polygon of double the number of sides by $p'$, and the similar circumscribed polygon by P$'$. Then it is plain that the space O$ad$ is the same part of $p$ that O$a$A is of P, that O$a$M is of $p'$, and that O$a$BM is of P$'$; for each of these spaces requires to be repeated the same number of times, to complete the several polygons to which they respectively belong. Hence then, and because magnitudes are as their like multiples,

it follows that whatever relations are shown to exist among these spaces will be true also of the respective polygons of which they form part. Now the right-angled triangles ODa, OAa, BMA are similar : the two first furnish the proportion OD : Oa : : Oa : OA, or, which is the same thing, OD : OM : : OM : OA;
and, consequently, since triangles of the same altitude are as their bases, it follows that

$$ODa : OMa :: OMa : OAa;$$

that is, the triangle OMa is a mean between ODa and OAa : consequently the polygon $p'$ is a mean proportional between the polygons $p$ and P.

Again, the similar triangles ODa, BMA give the proportion OD : Oa : : BM : BA, or, which is the same thing, OD : OM : : aB : BA; and, consequently, since triangles of the same altitude are as their bases, it follows that ODa : OMa : : OaB : OBA; therefore

$$ODa+OMa : 2\,ODa :: OaB+OBA : 2\,OaB.$$

Consequently $p+p' : 2p :: P : P'$.

*Scholium.* It was proved (B. V, Prop. XII), that a circle is equivalent to the rectangle contained by its radius and a straight line equivalent to half its circumference. In order, therefore, to construct a rectangle equivalent to any given circle, it would only be necessary, from having the radius, to draw a straight line equal to half the circumference. But this is a problem which has never yet been effected; so that the equivalent rectangle remains still undetermined, and therefore *the quadrature of the circle,* as this problem is called, is not capable of being rigorously ascertained. This, however, is a circumstance little to be regretted; for it has been shown (B. V, Prop. XI, Cor.) that polygons may be inscribed in and circumscribed about a circle, that shall approach so near to coincidence with it as to differ from it by a magnitude less than any that can be possibly assigned: a degree of approximation obviously equivalent to perfect accuracy, since no magnitude can be found sufficiently small to denote its difference therefrom. The principal object of inquiry then should be, at least in a practical point of view, how we may most expeditiously carry on the approximation alluded to; and the problem above furnishes us with one of the best elementary methods for this purpose that can be given.

Let us represent the radius of the circle by 1, and let the first inscribed and circumscribed polygons be squares: the side of the former will be $\sqrt{2}$, and that of the latter 2; so that the surface of the former will be 2, and that of the latter 4. Now it has been proved in the proposition that the surface of the inscribed octagon, or, as we have denoted it, $p'$, will be a mean proportional between the two squares $p$ and P, so that $p' = \sqrt{8} = 2 \cdot 8284271$. Also from the proportion $p+p' : 2p :: \mathrm{P} : \mathrm{P}'$, we obtain the numerical value of the circumscribed octagon; that is,

$$\mathrm{P}' = \frac{2p.\mathrm{P}}{p+p'} = \frac{16}{2+\sqrt{8}} = 3 \cdot 3137085.$$

Having thus obtained numerical expressions for the inscribed and circumscribed polygons of eight sides, we may, from these, by an application of the same two proportions in a similar way, determine the surfaces of those of sixteen sides, and thence the surfaces of polygons of thirty-two sides; and so on till we arrive at an inscribed and circumscribed polygon, differing from each other, and consequently from the circle, so little that either may be considered as equivalent to it. The subjoined table exhibits the *area*, or numerical expression for the surface of each succeeding polygon, carried to seven places of decimals.

| *Number of sides.* | *Area of the inscribed polygon.* | *Area of the circumscribed polygon.* |
|---|---|---|
| 4 | 2·0000000 | 4·0000000 |
| 8 | 2·8284271 | 3·3137085 |
| 16 | 3·0614674 | 3·1825979 |
| 32 | 3·1214451 | 3·1517249 |
| 64 | 3·1365485 | 3·1441184 |
| 128 | 3·1403311 | 3·1422236 |
| 256 | 3·1412772 | 3·1417504 |
| 512 | 3·1415138 | 3·1416321 |
| 1024 | 3·1415729 | 3·1416025 |
| 2048 | 3·1415877 | 3·1415951 |
| 4096 | 3·1415914 | 3·1415933 |
| 8192 | 3·1415923 | 3·1415928 |
| 16384 | 3·1415925 | 3·1415927 |
| 32768 | 3·1415926 | 3·1415926 |

It appears then that the inscribed and circumscribed polygons of 32768 sides differ so little from each other, that the numerical value of each, as far as seven places of decimals, is absolutely the same : and as the circle is between the two, it cannot, strictly speaking, differ from either so much as they do from each other; so that the number 3·1415926 expresses the area of a circle whose radius is 1, correctly as far as seven places of decimals. We may, therefore, conclude, that were the absolute quadrature of the circle attainable, it would exactly coincide with the above number as far at least as the seventh decimal place, which is an extent even beyond what the most delicate numerical calculations are ever likely to require.

Having found the numerical expression for the surface of a circle whose radius is 1, we readily find the area of any circle whatever; for since the surfaces are as the squares of the radii, we have only to multiply the square of the radius of any proposed circle by the number..... 3·14159, &c., and the product will be the area.

Also, since the surface of a circle is equivalent to half the circumference multipled by the radius (B. V, Prop. XII), it follows, that when the radius is 1, the half circumference must be 3·14159, &c.; or since the circumferences of circles are as their radii, when the diameter is 1, the circumference will be 3·14159, &c.; so that the circumference of any circle is found by multiplying its diameter by 3·14159, &c., or, as is usual, simply by 3·1416.

For ordinary purposes of mensuration, the circumference will be determined with sufficient precision by multiplying the diameter by 22, and dividing the product by 7, which is the approximation discovered by ARCHIMEDES. A still nearer method would be to multiply the diameter by 355, and divide the product by 113, which approximation was discovered by METIUS.

(93.) Were it necessary, this approximation might be continued to double the above number of decimals: it has indeed been carried by some to a much greater length than this. LUDOLPH VAN CEULEN had the patience to extend the approximation as far as the thirty-sixth place of decimals, by a method somewhat different indeed from that above described, but requiring an equal degree of labor and attention.

Since his time, the quadrature of the circle has been approached still nearer by other methods.

(94.) The following infinite series,

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x + \&c.,$$

where $x$ is an arc less than 90°, was discovered by GREGORY, and is the foundation of almost every effective method since employed in determining the ratio of the circumference to the diameter of the circle. It is not, in its simple form, well adapted to use, on account of its slow convergency; as it requires a great number of terms to be computed, in order to obtain a moderate degree of approximation.

When $x = 45°$, $\tan x = 1$, and Gregory's series becomes

$$\tfrac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \&c.,$$

or $$\tfrac{1}{4}\pi = \frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11} + \frac{2}{13.15} + \&c.;$$

where $\pi =$ the ratio sought.

When $x = 30° = \tfrac{1}{6}\pi$, we have $\tan x = \frac{1}{\sqrt{3}}$, and Gregory's series becomes

$$\tfrac{1}{6}\pi = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \&c.\right).$$

This is called HALLEY'S *series*: it also converges very slowly.

EULER made use of the following series, which converges with considerable rapidity:

$$\frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{3}\right) - \tfrac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \tfrac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) - \&c.$$

The celebrated series used by MACHIN, by which he carried the quadrature as far as the 100th place of decimals, is

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \frac{1}{7.5^7} + \&c.\right)$$
$$- \left(\frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \&c.\right).$$

Professor **William Rutherford**, of the Royal Military Academy, published in the Philosophical Transactions of the Royal Society of London, in 1841, his method of computing this ratio. His series is

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3.5^3} + \frac{1}{5.5^5} - \&c.\right) - \left(\frac{1}{70} - \frac{1}{3.70^3} + \frac{1}{5.70^5} - \&c.\right)$$
$$+ \left(\frac{1}{99} - \frac{1}{3.99^3} + \frac{1}{5.99^5} - \&c.\right)$$

The divisors 70 and 99, which enter into this series, are easily employed, from their being readily resolved into simple factors. In using the divisor 99, Mr. Rutherford employed synthetic division, thus greatly simplifying the labor. The series itself was derived from the equation

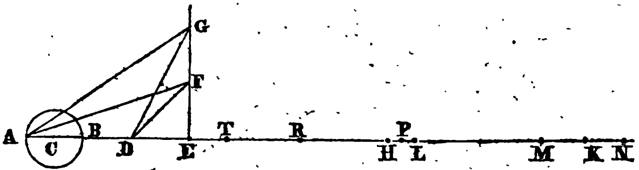
$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99},$$

which was first published by Euler in 1764.

Mr. Rutherford carried his approximation to 208 decimals, which value is

3·14159265358979323846264338327950288419716939937510582
09749445923078164062862089986280348253421170679821 48
08651328230664709384460955058223172535940812848473 78
13920386338302157473996008259312591294018328065 1744.

(95.) The following geometrical construction is very simple, and gives this ratio sufficiently accurate for all practical purposes:



Let ACB be the diameter of the given circle; produce it towards N; take BD and DE each equal to AB; through E draw EG perpendicular to AE, and take EF and FG each equal to AB; join AG, AF, DG and DF. Set off on the line EN, from E, the distances EH and HK, each equal to AG; then set off, in the opposite direction, the distance KL equal to AF, and from L set off LM equal to DG; also set off MN equal to DF. Then bisect EN at the point P; bisect EP at the point R, and finally trisect ER at the point T; then will CT be the circumference of the circle, nearly.

For, by construction, we have, if we call the diameter a unit,..... $CE = 2\frac{1}{2}$; $EL = 2EH - KL = 2\sqrt{13} - \sqrt{10}$; $LM = \sqrt{5}$; $MN = \sqrt{2}$. Therefore $EN = 2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2}$; and

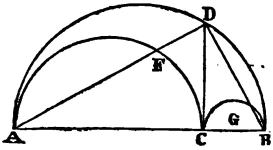
$$ET = \tfrac{1}{12}(2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2}),\text{ and therefore}$$

$$CT = 2\tfrac{1}{2} + \tfrac{1}{12}(2\sqrt{13} - \sqrt{10} + \sqrt{5} + \sqrt{2}) = 3{\cdot}1415922,\ \&c.,$$

which is the ratio true to six decimals. For simplicity and accuracy, a better graphic method of finding this ratio can hardly be expected or even desired.

(96.) THEOREM. *If a straight line be divided into two parts, the semicircle described on the whole line as a diameter, will be equal to the sum of the semicircles described on the two parts, together with the circle described on a line which is a mean proportional between the two parts.*

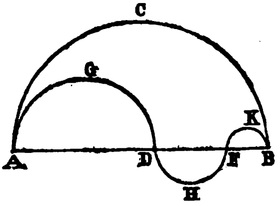
Let AB be a straight line divided into the two parts AC & CB; then will the semicircle ADB be equal to the sum of the semicircles AFC and CGB, together with a circle described on CD, which is a mean proportional between AC and CB.



Since the areas of circles, and consequently of semicircles, are to each other as their radii or diameters (B. V, Prop. XIII), we have the semicircle on AD equal to the sum of the semicircles on AC and CD; also the semicircle on BD equal to the sum of the semicircles on CB and CD (B. IV, Prop. VIII): therefore the sum of the semicircles on AD and BD, which is equal to the semicircle on AB, is equal to the sum of the semicircles on AC and CB, together with the circle on CD.

(97.) THEOREM. *If a straight line be divided into any number of parts, the semicircumference described on the whole line will be equal to the sum of the semicircumferences described on the parts.*

Let AB be a straight line divided at the points D & F; then will the semicircumference ACB be equal to the sum of the semicircumferences AGD, DHF and FKB.



For, circumferences are as their diameters, and consequently semicircumferences must also be as their diameters; therefore we have

ACB : AGD : : AB : AD,

ACB : DHF : : AB : DF, and

ACB : FKB : : AB : FB. Consequently,

ACB : AGD+DHF+FKB : AB : : AD+DF+FB.

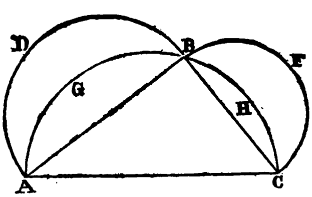
Since the third term of this proportion is equal to the fourth term, it follows that the first term is equal to the second term : hence,

ACB = AGD + DHF + FKB.

(98.) THEOREM. *If, on the sides of a triangle inscribed in a semicircle, semicircles be described, the two lunes formed thereby will together be equal to the area of the triangle.*

Let ABC be a triangle inscribed in a semicircle.

On AB, BC, let semicircles ADB, BFC be described; the lunes ADBG, BFCH are together equal to the triangle ABC.

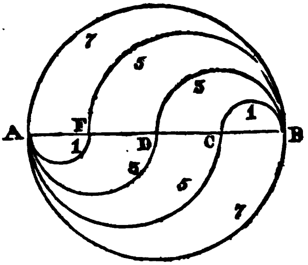


Since the areas of circles, and of course semicircles, are as the squares of their radii or diameters (B. V, Prop. XIII), we have

semicircle ABC : ADB : : $AC^2$ : $AB^2$, and
semicircle ABC : BFC : : $AC^2$ : $BC^2$; therefore,
ABC : ADB+BFC : : $AC^2$ : $AB^2+BC^2$.

But, since the triangle ABC is right-angled, $AC^2=AB^2+BC^2$; therefore ABC = ADB+BFC. From these equals take away the segments AGB, BHC, and we shall have the triangle ABC = the sum of the lunes ADBG and BFCH.

(99.) **Problem.** *To divide a given circle into any number of parts, which shall be equal in area and equal in perimeter, and not have the portions in the form of sectors.*

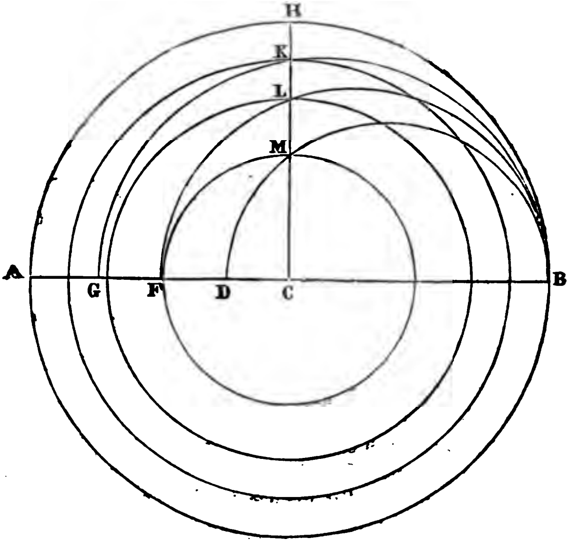


Let AB be the diameter of the circle, and suppose we wish to divide it into four parts. Divide the diameter into four equal parts at the points C, D, F. Then describe the semicircles upon the opposite sides of the different segments of the diameter, as exhibited in the diagram. Now if we suppose the diameter to be effaced, the four portions will fulfil the conditions required.

Semicircles are to each other as the squares of their diameters (B. V, Prop. XIII). Representing the semicircle described on BC as a diameter by 1, it follows that the one described on the diameter BD will be represented by 4, the one described on BF will be represented by 9, while the entire semicircle described on AB will be denoted by 16 : hence the spaces above the diameter AB, as well as those below, will be to each other as the numbers 1, 3, 5, 7. Now when the diameter is supposed to be removed, the portions which are thus united will, in each case, be represented by 8, and therefore they are all equal.

The sum of the semicircumferences described on BC and CA is equal to the semicircumference on AB (Art. 97), and the same for the semicircumferences described on the other segments into which the diameter AB is divided; hence each portion has for its perimeter an entire circumference.

(100.) Problem. *It is required to find what part of the diameter of a grindstone a given number of individuals must respectively grind off, so that they shall receive equal portions of the stone.*



Let AB be the diameter of the stone, and suppose it is to be equally shared among four individuals.

Divide the radius AC into four equal parts; then upon BD, BF, BG, describe semicircumferences. Draw the radius CH at right angles to the diameter AB; and the points K, L, M, thus determined, will be the points to which they must grind respectively.

We have (B. IV, Prop. XVIII, Cor. 1)

$$\begin{aligned} CM^2 &= BC \times CD, \\ CL^2 &= BC \times CF, \\ CK^2 &= BC \times CG, \text{ and} \\ CH^2 &= BC \times CA; \end{aligned}$$

therefore $CM^2$, $CL^2$, $CK^2$, $CH^2$ are to each other as CD, CF, CG, CA, or as 1, 2, 3, 4. Hence the areas of the circles whose radii are CM, CL, CK, CH are to each other as 1, 2, 3, 4 (B. V, Prop. XIII); therefore the stone has in this way been divided equally among four individuals. A similar method would apply for a greater number of divisions.

If we denote the radius of the stone by R, and the number of individuals by $n$, we shall have

$$BC = R; \quad CD = \frac{R}{n}; \quad CF = \frac{2R}{n}; \quad CG = \frac{3R}{n}, \text{ \&c.}$$

Hence

$$\begin{aligned} CM &= \sqrt{R \times \frac{R}{n}} = R\sqrt{\frac{1}{n}}; \\ CL &= \sqrt{R \times \frac{2R}{n}} = R\sqrt{\frac{2}{n}}; \\ CK &= \sqrt{R \times \frac{3R}{n}} = R\sqrt{\frac{3}{n}}, \\ &\text{\&c.} \quad \text{\&c.} \quad \text{\&c.} \end{aligned}$$

# BOOK SIXTH.

## DEFINITIONS.

1. The *common section* of two planes is the line in which they meet to cut each other.

2. A line is *perpendicular to a plane,* when it is perpendicular to any two lines in that plane which meet it.

3. One *plane is perpendicular* to another, when every line in the one which is perpendicular to their common section is perpendicular to the other plane.

4. The *inclination of two planes to each other,* or *the angle they form between them,* is the angle contained by two lines drawn from any point in the common section, and at right angles to the same, one of these lines in each plane.

5. A line is *parallel to a plane,* when, if both are produced to any distance, they do not meet; and, conversely, the plane is then also parallel to the line.

6. Two *planes are parallel to each other,* when, both being produced to any distance, they do not meet.

7. A *solid angle* is the angular space included between three or more planes which meet at the same point.

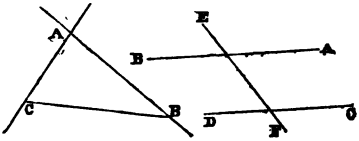
### PROPOSITION I.

THEOREM. *One part of a straight line cannot be in a plane, and another part out of it.*

For (B. I, Def. VII), when a straight line has two points common with a plane, it lies wholly in that plane.

### PROPOSITION II.

THEOREM. *Two straight lines which intersect each other, lie in the same plane, and determine its position.*



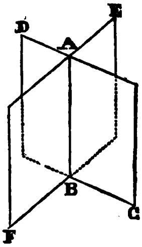
Let AB, AC be two straight lines which intersect each other in A; and conceive some plane passing through one of the lines as AB, and if also AC be in this plane, then it is clear that the two lines, according to the terms of the proposition, are in the same plane; but if not, let the plane passing through AB be supposed to be turned round AB till it passes through the point C, then the line AC, which has two of its points A and C in this plane, lies wholly in it; and hence the position of the plane is determined by the single condition of containing the two straight lines AB, AC.

*Cor.* 1. A triangle ABC, or any three points not in a straight line, determines the position of a plane.

*Cor.* 2. Hence, also, two parallels AB, CD determine the position of a plane; for, drawing the secant EF, the plane of the two straight lines AB, EF is that of the parallels AB, CD.

### PROPOSITION III.

**Theorem.** *The common section of two planes is a straight line.*

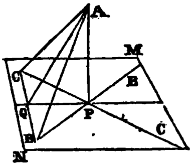


Let DC and EF be two planes cutting each other, and A, B two points in which the planes meet. Draw the line AB; this line is the common intersection of the two planes.

For, because the straight line touches the two planes in the points A and B, it lies wholly in both these planes, or is common to both of them; that is, the common intersection of the two planes is in a straight line.

### PROPOSITION IV.

**Theorem.** *If a straight line is perpendicular to each of two straight lines, at their point of intersection, it will be perpendicular to the plane of these lines.*



Let AP be perpendicular to the two lines PB, PC, at their point of intersection P; then will it be perpendicular to MN the plane of the lines.

Through P, draw in the plane MN any line as PQ; and through any point of this line, as Q, draw BQC, so that BQ = QC (B. IV, Prop. VI): join AB, AQ, AC.

The base BC being divided into two equal parts at the point Q, the triangle BPC (B. II, Prop. XII) will give

$$PC^2 + PB^2 = 2PQ^2 + 2QC^2.$$

The triangle BAC will, in like manner, give

$$AC^2 + AB^2 = 2AQ^2 + 2QC^2.$$

Taking the first equation from the second, and observing that the triangles APC, APB, which are both right-angled at P, give $AC^2 - PC^2 = AP^2$, and $AB^2 - PB^2 = AP^2$, we shall have $2AP^2 = 2AQ^2 - 2PQ^2$,

or $AP^2 = AQ^2 - PQ^2$;

that is, $AQ^2 = AP^2 + PQ^2$.

Hence the triangle APQ is right-angled at P, and therefore AP is perpendicular to PQ.

*Scholium.* Thus it is evident, not only that a straight line may be perpendicular to all the straight lines which pass through its foot in a plane, but that it always must be so, whenever it is perpendicular to two straight lines drawn in the plane.

*Cor.* 1. The perpendicular AP is shorter than any oblique line AQ; therefore it measures the true distance from the point A to the plane MN.

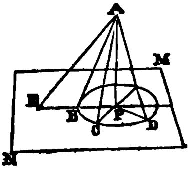
*Cor.* 2. At a given point P on a plane, it is impossible to erect more than one perpendicular to that plane. For, if there could be two perpendiculars at the same point P, draw along these two perpendiculars a plane, whose intersection with the plane MN is PQ; then those two perpendiculars would be perpendicular to the line PQ at the same point, and in the same plane, which is impossible.

It is also impossible to draw, from a given point out of a plane, two perpendiculars to that plane. For, let AP, AQ, be these two perpendiculars; then the triangle APQ would have two right angles APQ, AQP, which is impossible.

## PROPOSITION V.

THEOREM. *Oblique lines equally distant from the perpendicular to a plane, are equal; and, of two oblique lines unequally distant from the perpendicular, that which is nearer is less than that more remote.*

For the angles APB, APC, APD being right, if we suppose the distances PB, PC, PD to be equal to each other, the triangles APB, APC, APD will have each an equal angle contained by equal sides, and therefore they will be equal; therefore the hypothenuses, or the oblique lines AB, AC, AD will be equal to each other. In like manner, if the distance PE be greater than PD or its equal PB, the oblique line AE will evidently be greater than AB, or its equal AD; that is, AB will be less than AE.

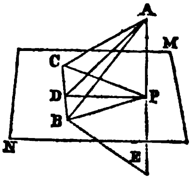


*Cor.* All the equal oblique lines AB, AC, AD, &c., terminate in the circumference of a circle BCD, described from P the foot of the perpendicular as a centre. Therefore a point A being given out of a plane, the point P at which the perpendicular drawn from A would meet that plane, may be found by marking upon that plane three points B, C, D, equally distant from the point A, and then finding the centre of the circle which passes through these points : this centre will be P the point sought.

*Schol.* The angle ABP is called *the inclination of the oblique line* AB *to the plane* MN; which inclination is equal with respect to all such lines AB, AC, AD, as are equally distant from the perpendicular; for all the triangles ABP, ACP, ADP, &c. are equal to each other.

### PROPOSITION VI.

THEOREM. *If, from a point without a plane, a perpendicular be drawn to that plane, and from the foot of the perpendicular a line be drawn perpendicular to a line in the plane, and from the point of intersection a line be drawn to the first point, this last line will be perpendicular to the line in the plane.*



Let AP be a line perpendicular to the plane MN, and PD perpendicular to BC; then will AD be perpendicular to BC.

Take DB = DC, and join PB, PC, AB, AC. Since DB = DC, the two right-angled triangles PDB, PDC are equal, and PB = PC; and with regard to the perpendicular AP, since PB = PC, the oblique line AB = AC (B. VI, Prop. V); therefore the two triangles ADB, ADC have the three sides of the one equal to the three sides of the other; consequently they are equal (B. I, Prop. VIII), and the angle ADB is equal to ADC; therefore each is a right angle, and AD is perpendicular to BC (Def. X).

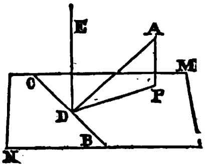
*Cor.* It is evident, likewise, that BC is perpendicular to the plane APD; since BC is at once perpendicular to the two straight lines AD, PD (B. VI, Prop. IV).

*Schol.* The two straight lines AE, BC afford an instance of two lines which do not meet, because they are not situated in the same plane. The shortest distance between these lines is the straight line PD, which is perpendicular both to the line AP and to the line BC. The distance PD is the shortest between these two lines; for if we join any other two points, such as A and B, we shall have AB > AD, AD > PD; therefore AB > PD.

The two lines AE, CB, though not situated in the same plane, are conceived as forming a right angle with each other, because AD and the line drawn through one of its points parallel to BC would make with each other a right angle. In the same manner, the line AB and the line PD, which represent any two straight lines not situated in the same plane, are supposed to form with each other the same angle which would be formed by AB and a straight line parallel to PD drawn through one of the points of AB.

### PROPOSITION VII.

THEOREM. *If one of two parallel lines is perpendicular to a plane, the other will also be perpendicular to this plane.*



Let AP & ED be parallel lines, of which AP is perpendicular to the plane MN; then will ED be also perpendicular to this plane.

Along the parallels AP, DE, extend a plane; its intersection with the plane MN will be PD. In the plane MN draw BC perpendicular to PD, and join AD.

Then BC is perpendicular to the plane APDE (B. VI, Prop. VI, Cor.), and therefore the angle BDE is right. But the angle EDP is right also, since AP is perpendicular to PD, and DE parallel to AP; therefore the line DE is perpendicular to the two straight lines DP, DB; therefore it is perpendicular to their plane MN.

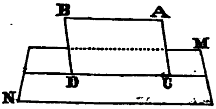
*Cor.* 1. Conversely, if the straight lines AP, DE are perpendicular to the same plane MN, they will be parallel. For, if they be not so, draw through the point D a line parallel to AP : this parallel will be perpendicular to the plane MN ; therefore through the same point D more than one perpendicular might be drawn to the same plane, which (B. VI, Prop. IV, Cor. 2) is impossible.

*Cor.* 2. Two lines A and B, parallel to a third line C, are parallel to each other. For, conceive a plane perpendicular to the line C : the lines A and B, being parallel to C, will be perpendicular to the same plane ; therefore, by the preceding corollary, they will be parallel to each other.

When the three lines are in the same plane, the case falls under Book I, Prop. XXI.

### PROPOSITION VIII.

**Theorem.** *If a straight line without a plane is parallel to a line within the plane, it is parallel to the plane itself.*

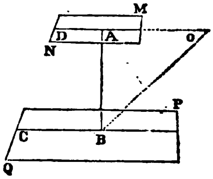


Let the straight line AB, without the plane MN, be parallel to the line CD of this plane ; then will AB be parallel to the plane MN.

For if the line AB, which lies in the plane ABDC, could meet the plane MN, this could only be in some point of the line CD, the common intersection of the two planes ; but AB cannot meet CD, since they are parallel ; hence it will not meet the plane MN ; therefore (Def. 5) it is parallel to that plane.

### PROPOSITION IX.

**Theorem.** *If two planes are perpendicular to the same line, they are parallel.*

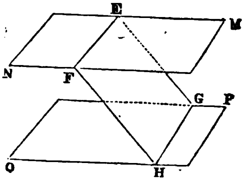


Let the planes MN and PQ be each perpendicular to AB; then will they be parallel.

For, if they can meet anywhere, let O be one of their common points, and join OA, OB. The line AB, which is perpendicular to the plane MN, is perpendicular to the straight line OA drawn through its foot in that plane. For the same reason, AB is perpendicular to BO. Therefore OA and OB are two perpendiculars drawn from the same point O, upon the same straight line, which is impossible; therefore the planes MN, PQ cannot meet each other, and consequently they are parallel.

### PROPOSITION X.

**Theorem.** *The intersection of two parallel planes with a third plane will be parallel.*

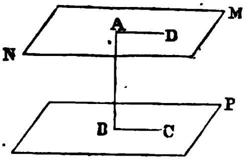


Let the two parallel planes MN and PQ intersect the plane EH; then will EF be parallel to HG.

For, if the lines EF, GH, lying in the same plane, were not parallel, they would meet each other when produced; therefore the planes MN, PQ, in which those lines are situated, would also meet; therefore the planes would not be parallel.

### PROPOSITION XI.

THEOREM. *A line which is perpendicular to one of two parallel planes, is perpendicular to the other also.*

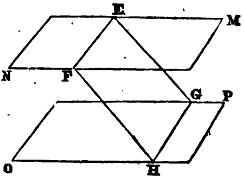


Let the two planes MN & PQ be parallel; then if the line AB is perpendicular to the plane MN, it will also be perpendicular to PQ.

Having drawn any line BC in the plane PQ, along the lines AB and BC extend a plane ABC, intersecting the plane MN in AD; the intersection AD will be parallel to BC (B. VI, Prop. x). But the line AB, being perpendicular to the plane MN, is perpendicular to the straight line AD; therefore also to its parallel BC. Hence the line AB, being perpendicular to any line BC drawn through its foot in the plane PQ, is consequently perpendicular to that plane.

### PROPOSITION XII.

THEOREM. *Two parallel lines, included between two parallel planes, are equal.*



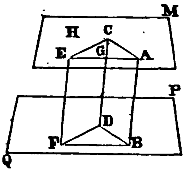
Let EG, FH be two parallel lines included between the two parallel planes MN, PQ; then will these lines be equal.

Through the parallels EG, FH, draw the plane EGHF to meet the parallel planes in EF and GH. The intersections EF, GH (B. VI, Prop. x) are parallel to each other; so likewise are EG, FH : therefore the figure EGHF is a parallelogram, and EG = FH.

*Cor.* Hence it follows that *two parallel planes are every where equidistant*; for if EG and FH are perpendicular to the two planes MN, PQ, they will be parallel to each other (B. VI, Prop. VII, Cor. 1), and therefore equal.

### PROPOSITION XIII.

**Theorem.** *If two angles, not situated in the same plane, have their sides parallel and lying in the same direction, they will be equal, and the planes in which they are situated will be parallel.*



Let CAE, DBF be two angles not situated in the same plane, having AC parallel to BD and lying in the same direction, and AE parallel to BF and also lying in the same direction; then will these angles be equal, and their planes will be parallel.

Make AC = BD, AE = BF; and join CE, DF, AB, CD, EF. Since AC is equal and parallel to BD, the figure ABDC is a parallelogram; therefore CD is equal and parallel to AB. For a similar reason, EF is equal and parallel to AB; hence also CD is equal and parallel to EF. The figure CEFD is therefore a parallelogram, and the side CE is equal and parallel to DF; therefore the triangles CAE, DBF have their corresponding sides equal, and consequently the angle CAE = DBF.

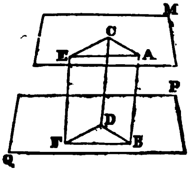
Again, the plane ACE is parallel to the plane BDF. For, suppose the plane parallel to BDF, drawn through the point A, were to meet the lines CD, EF in points different from C and E, for instance in G and H; then (B. VI, Prop. XII) the three lines AB, GD, HF would be equal. But the lines AB, CD, EF are already known to be equal; hence CD = GD, and HF = EF, which is absurd: hence the plane ACE is parallel to BDF.

*Cor.* If two parallel planes MN, PQ are met by two other planes CABD, EABF, the angles CAE, DBF formed by the intersections of the parallel planes will be equal; for (B. VI, Prop. X) the intersection AC is parallel to BD, and AE to BF, and therefore the angle CAE = DBF.

### PROPOSITION XIV.

THEOREM. *If three straight lines, not situated in the same plane, are equal and parallel, the triangles formed by joining their corresponding opposite extremities will be equal, and their planes will be parallel.*

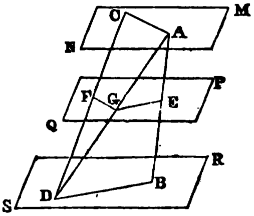
Let ACE, BDF be two triangles formed by joining the opposite extremities of the three equal and parallel lines AB, CD, EF; then will these triangles be equal, and their planes will be parallel.



For, since AB is equal and parallel to CD, the figure ABDC is a parallelogram; hence the side AC is equal and parallel to BD. For a like reason, the sides AE, BF are equal and parallel; as also CE, DF. Therefore the two triangles ACE, BDF are equal; and, consequently, as in the last proposition, their planes are parallel.

### PROPOSITION XV.

THEOREM. *Two straight lines, included between three parallel planes, are cut proportionally.*



Suppose the line AB to meet the parallel planes MN, PQ, RS at the points A, E, B; and the line CD to meet the same planes at the points C, F, D : then

AE : EB : : CF : FD.

Draw AD meeting the plane PQ in G, and join AC, EG, GF, BD. The intersections EG, BD of the parallel planes PQ, RS in the plane ABD, are parallel (B. VI, Prop. x); therefore

AE : EB : : AG : GD. [B. IV, Prop. I.]

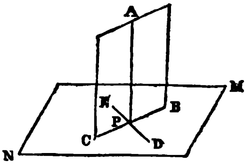
In like manner, the intersections AC, GF being parallel, AG : GD : : CF : FD.

The ratio AG : GD is the same in both; hence

AE : EB : : CF : FD.

### PROPOSITION XVI.

THEOREM. *If a straight line is perpendicular to a plane, then every plane passing through this line will also be perpendicular to the first plane.*



Let the line AP be perpendicular to the plane MN; then, any plane, as AB, passing through this line, will also be perpendicular to the plane MN.

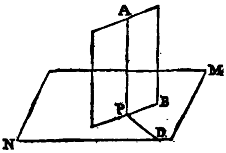
For, let BC be the intersection

of the planes AB, MN. In the plane MN draw DE perpendicular to BP; then the line AP, being perpendicular to the plane MN, will be perpendicular to each of the two straight lines BC, DE. But the angle APD, formed by the two perpendiculars PA, PD at their common intersection BP, is the measure of the angle of the two planes (B. VI, Def. 4); and since in the present case the angle is a right angle, the two planes are perpendicular to each other.

*Scholium.* When the three lines such as AP, BP, DP are perpendicular to each other, each of these lines is perpendicular to the plane of the other two, and the planes themselves are perpendicular to each other.

### PROPOSITION XVII.

THEOREM. *If two planes are perpendicular to each other, every line drawn in one of them perpendicular to their common intersection, will be perpendicular to the other plane.*



Let the planes AB, MN be perpendicular to each other; and in the plane AB, let PA be drawn perpendicular to the common intersection PB; then will it be perpendicular to the plane MN.

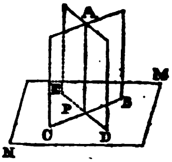
For, in the plane MN draw PD perpendicular to PB; then because the planes are perpendicular, the angle APD is a right angle: therefore the line AP is perpendicular to the two straight lines PB, PD, and consequently perpendicular to their plane MN.

*Cor.* If the plane AB be perpendicular to the plane MN, and if at a point P of the common intersection a perpendicular be drawn to the plane MN, that perpendicular will be in the plane AB. For, if not, then in the plane AB we might draw AP perpendicular to PB their common intersection, and this AP at the same time would be perpendicular to the plane MN; therefore at the same point P there would be two perpendiculars to the plane MN, which is impossible.

### PROPOSITION XVIII.

THEOREM. *If two planes be perpendicular to a third plane, their common intersection will be also perpendicular to the third plane.*

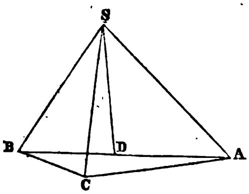
Let AB, AD be perpendicular to MN; then will their common intersection AP be perpendicular to the same plane MN.



For, at the point P draw the perpendicular to the plane MN; then that perpendicular must be in the plane AD, as also in AB (by the last proposition); therefore it is their common intersection AP.

### PROPOSITION XIX.

THEOREM. *If a solid angle is formed by three plane angles, the sum of any two of these angles will be greater than the third.*



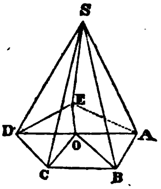
The proposition requires demonstration only when the plane angle, which is compared to the sum of the other two, is greater than either of them. Therefore suppose the solid angle S to be formed by three plane angles ASB, ASC, BSC, whereof the angle ASB is the greatest; we are to show that ASB < ASC + BSC.

In the plane ASB make the angle BSD = BSC; draw the straight line ADB at pleasure; and, having taken SC = SD, join AC, BC.

The two sides BS, SD are equal to the two BS, SC; the angle BSD=BSC; therefore the triangles BSD, BSC are equal, and BD=BC. But AB < AC + BC : taking BD from the one side, and from the other its equal BC, there remains AD < AC. The two sides AS, SD are equal respectively to the two AS, SC; the third side AD is less than the third side AC; therefore the angle ASD<ASC. Adding BSD = BSC, we shall have ASD + BSD or ASB < ASC + BSC.

### PROPOSITION XX.

**Theorem.** *The sum of the plane angles which form a solid angle, is always less than four right angles.*



Conceive the solid angle S to be cut by any plane ABCDE : from O a point in that plane, draw to the several angles straight lines AO, OB, OC, OD, OE.

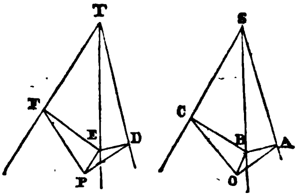
The sum of the angles of the triangles ASB, BSC, &c. formed about the vertex S, is equivalent to the sum of the angles of an equal number of triangles AOB, BOC, &c. formed about the point O. But at the point B the angles ABO, OBC taken together, make the angle ABC (B. VI, Prop. xviii) less than the sum of the angles ABS, SBC. In the same manner, at the point C we have $BCO + OCD < BCS + SCD$; and so with all the angles of the polygon ABCDE. Whence it follows that the sum of all the angles at the bases of the triangles whose common vertex is in O, is less than the sum of all the angles at the bases of the triangles whose common vertex is in S : hence, to make up the deficiency, the sum of the angles formed about the point O is greater than the sum of the angles about the point S. But the sum of the angles about the point O is equal to four right angles (B. I, Prop. i, Cor. 3); therefore the sum of the plane angles, which form the solid angle S, is less than four right angles.

*Schol.* This demonstration is founded on the supposition that the solid angle is convex, or that the plane of no one surface produced can ever meet the solid angle. If it were otherwise, the sum of the plane angles would no longer be limited, and might be of any magnitude.

### PROPOSITION XXI.

THEOREM. *If two solid angles are composed of three plane angles respectively equal to each other, the planes which contain the equal angles will be equally inclined to each other.*

Let the angle $ASC = DTF$; the angle $ASB = DTE$; and the angle $BSC = ETF$; then will the inclination of the planes ASC, ASB be equal to that of the planes DTF, DTE.



Having taken SB at pleasure, draw BO perpendicular to the plane ASC; from the point O at which that perpendicular meets the plane, draw OA, OC perpendicular to SA, SC; join AB, BC; next take $TE = SB$; draw EP perpendicular to the plane DTF; from the point P draw PD, PF perpendicular to TD, TF; lastly, join DE, EF.

The triangle SAB is right-angled at A, and the triangle TDE at D; and since the angle $ASB = DTE$, we have $SBA = TED$. Likewise $SB = TE$; therefore the triangle SAB is equal to the triangle TDE; therefore $SA = TD$, and $AB = DE$. In like manner it may be shown that $SC = TF$, and $BC = EF$. That granted, the quadrilateral

SAOC is equal to the quadrilateral TDPF; for, place the angle ASC upon its equal DTF; because SA=TD, and SC=TF, the point A will coincide with D, and the point C with F; and at the same time AO, which is perpendicular to SA, will coincide with PD which is perpendicular to TD, and in like manner CO with FP; wherefore the point O will coincide with the point P, and AO will be equal to DP. But the triangles AOB, DPE are right-angled at O and P; the hypothenuse AB = DE, and the side AO = DP: hence those triangles are equal; therefore the angle AOB = PDE. The angle OAB is the inclination of the two planes ASB, ASC, and the angle PDE is that of the two planes DTE, DTF: hence these two inclinations are equal to each other.

It must, however, be observed, that the angle A of the right-angled triangle AOB is properly the inclination of the two planes ASB, ASC, only when the perpendicular BO falls on the same side of SA as SC falls; for if it fell on the other side, the angle of the two planes would be obtuse, and, added to the angle A of the triangle OAB, it would make two right angles. But, in the same case, the angle of the two planes TDE, TDF would also be obtuse, and, added to the angle D of the triangle PDE, it would make two right angles; and the angle A being thus always equal to the angle at D, it would follow, in the same manner, that the inclination of the two planes ASB, ASC must be equal to that of the two planes DTE, DTF.

*Scholium relative to the measure of solid angles.* A more general definition of solid angles than that given at the commencement of this book, is, that a solid angle is the angular space included between several plane surfaces, or one or more curved surface meeting in the point which forms the summit of the angle.

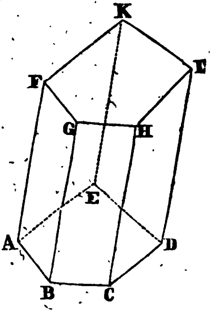
According to this definition, solid angles bear just the same relation to the surfaces which comprise them, as plane angles do to the lines by which they are included; so that, as in the latter, it is not the magnitude of the lines, but their mutual inclination which determines the angles; so, in the former, it is not the magnitude of the planes, but their mutual inclination which determines the solid angles. According to this view of the subject, the spherical surface described about the summit of any solid angle as a centre, will become a measure of that angle; as the circular arc is employed to measure and to compare rectilinear angles. Let us imagine, in the first place, such a sphere to be described about any given solid angle comprised under three plane angles, and that those planes are produced till they cut the surface of the sphere; then will the surface of the spherical triangle included between those planes be the measure, or may be assumed as the measure, of the solid angle made by the planes at the common point of meeting; for no change can be conceived in the relative position of the bounding planes, that is, in the magnitude of the solid angle, without a corresponding and proportional mutation in the surface of the spherical triangle. And if, in like manner, the three or more plane surfaces comprising another solid angle be produced till they cut the surface of the same, or of an equal sphere, whose centre coincides with the summit of the angle, the surface of the spherical triangle or polygon included between the planes which determine the angle, will, in like manner, be a correct measure of that angle; and the ratio which subsists between the areas of these triangles and polygons, or other surfaces thus formed, will be accurately the ratio which subsists between the solid angles constituted by the meeting of the several planes or surfaces at the centre of the sphere.

It will, of course, be understood, that this measurement has only a relation to the magnitude of the angles. It has no reference to their geometrical properties, which may be very different, although their magnitudes, as above estimated, may be the same.

# BOOK SEVENTH.

## DEFINITIONS.

1. A *prism* is a solid contained by plane figures, of which two that are opposite are equal, similar, and parallel to one another; and the others are parallelograms. To construct this solid, let ABCDE be any rectilineal figure. In a plane parallel to ABC, draw the lines FG, GH, HI, &c. parallel to the sides AB, BC, CD, &c.; thus there will be formed a figure FGHIK, similar to ABCDE. Now let the vertices of the corresponding angles be joined by the lines AF, BG, CH, &c.; the faces ABGF, BCHG, &c. will evidently be parallelograms, and the solid thus formed will be a prism.



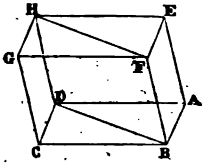
2. The equal and parallel plane figures ABCDE, FGHIK are called the *bases* of the prism. The other planes or parallelograms, taken together, constitute the *lateral* or *convex* surface of the prism.

3. The *altitude* of a prism is the perpendicular distance between its bases; and its *length* is a line equal to any one of its lateral edges, as AF or BG, &c.

4. A *right prism* is one in which the lateral edges AF, BG, &c. are perpendicular to the planes of its bases; then each of them is equal to the altitude of the prism : in every other case, the prism is oblique.

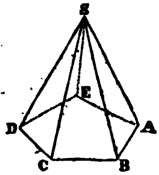
5. A prism is *triangular*, *quadrangular*, *pentagonal*, etc., according as the base is a triangle, a quadrilateral, a pentagon, etc.

6. A prism which has a parallelogram for its base, has all its faces parallelograms, and is called a *parallelopipedon*. A parallelopipedon is *rectangular*, when all its faces are rectangles.



7. When the faces of a rectangular parallelopipedon are square, it is called a *cube*.

8. A *pyramid* is a solid formed by several triangular planes which meet in a point, as S, and terminate in the same plane rectilineal figure ABCDE.



The plane figure ABCDE is called the *base* of the pyramid; the point S is its *vertex*; and the triangles ASB, BSC, &c. taken together, form the *convex* or *lateral* surface of the pyramid.

9. The *altitude* of a pyramid is the perpendicular drawn from the vertex to the plane of its base, produced if necessary.

10. A pyramid is *triangular*, *quadrangular*, etc., according as its base is a *triangle*, a *quadrangle*, etc.

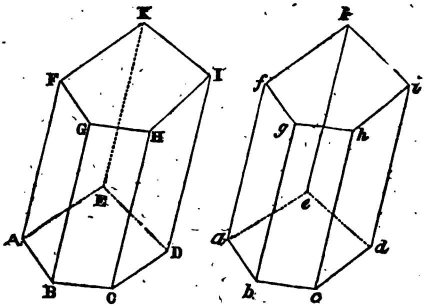
11. A pyramid is *regular*, when its base is a regular figure, and the perpendicular from its vertex passes through

the centre of its base; that is, through the centre of a circle which may be conceived to circumscribe its base.

12. Two solids are *similar*, when they are contained by the same number of similar planes, similarly situated, and having like inclinations to one another.

### PROPOSITION I.

THEOREM. *Two prisms are equal, when a solid angle in each is contained by three planes, which are equal in both, and similarly situated.*



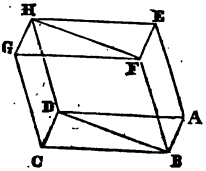
Let the base ABCDE be equal to the base *abcde;* the parallelogram ABGF equal to the parallelogram *abgf*, and the parallelogram BCHG equal to the parallelogram *bchg* : then will the prism ABCI be equal to the prism *abci*.

For, apply the base ABCDE upon its equal *abcde*, so that the bases (being equal) may coincide. But the three plane angles which form the solid angle B are respectively equal to the three plane angles which form the solid angle *b*; that is, ABC = *abc*, ABG = *abg*, and GBC = *gbc*, and they are also similarly situated; therefore the solid angles B and C are equal (B. VI, Prop. XXI), and therefore BG will coincide with its equal *bg*. And it is likewise evident, because the parallelograms ABGF and *abgf* are equal, that the side GF will coincide with its equal *gf*, and in the same manner GH with *gh*; therefore the upper base FGHIK will coincide with its equal *fghik*, and the two solids will be identical, since their vertices are the same.

*Cor. Two right prisms, which have equal bases and equal altitudes, are equal.* For, since the side AB is equal to *ab*, and the altitude BG to *bg*, the rectangle ABGF will be equal to *abgf*, and, in the same way, the rectangle BGHC will be equal to *bghc*; and thus the three planes which form the solid angle B will be equal to the three planes which form the solid angle *b*: hence the two prisms are equal.

### PROPOSITION II.

THEOREM. *In every parallelopipedon, the opposite planes are equal and parallel.*



By the definition of this solid, the bases ABCD, EFGH are equal parallelograms, and their sides are parallel: it remains only to show that the same is true of any two opposite lateral faces, such as

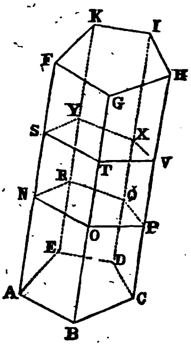
AEHD, BFGC. Now AD is equal and parallel to BC, because the figure ABCD is a parallelogram; for a like reason, AE is parallel to BF : hence the angle DAE is equal to the angle CBF, and the planes DAE, CBF are parallel; hence also the parallelogram DAEH is equal to the parallelogram CBFG. In the same way, it may be shown that the opposite parallelograms ABFE, DCGH are equal and parallel.

*Cor.* Since the parallelopipedon is a solid bounded by six planes, whereof those lying opposite to each other are equal and parallel, it follows that any face and the one opposite to it may be assumed as the bases of the parallelopipedon.

*Scholium.* If three straight lines AB, AE, AD, passing through the same point A, and making given angles with each other, are known, a parallelopipedon may be formed on those lines. For this purpose, a plane must be extended through the extremity of each line, and parallel to the plane of the other two; that is, through the point B a plane parallel to DAE, through D a plane parallel to BAE, and through E a plane parallel to BAD. The mutual intersections of these planes will form the parallelopipedon required.

### PROPOSITION III.

**Theorem.** *In every prism, the sections formed by parallel planes are equal polygons.*



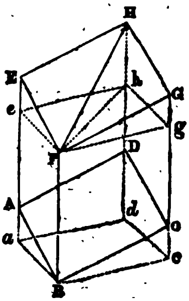
In the prism ABCI, let the sections NOPQR, STVXY be formed by parallel planes; then will these sections be equal polygons.

For the sides ST, NO are parallel, being the intersections of two parallel planes with a third plane ABGF; moreover the sides ST, NO are included between the parallels NS, OT, which are sides of the prism: hence NO is equal to ST. For like reasons, the sides OP, PQ, QR, &c. of the section NOPQR, are respectively equal to the sides TV, VX, XY, &c. of the section STVXY; and since the equal sides are at the same time parallel, it follows that the angles NOP, OPQ, &c. of the first section are respectively equal to the angles STV, TVX, &c. of the second: hence the two sections NOPQR, STVXY are equal polygons.

*Cor.* Every section in a prism, if made parallel to the base, is also equal to that base.

### PROPOSITION IV.

THEOREM. *If a plane be made to pass through the diagonal and opposite edges of a parallelopipedon, so as to divide it into two triangular prisms, those prisms are equal.*



Let the parallelopipedon ABCG be divided by the plane BDHF into the two triangular prisms ABDHEF, BCDFGH; then will those prisms be equal.

Through the vertices B and F, draw the planes B*adc*, F*ehg*, at right angles to the side BF, and meeting AE, DH, CG, the three other sides of the parallelopipedon, in the points *a*, *d*, *c* towards one direction, and in *e*, *h*, *g* towards the other: then the sections B*adc*, F*ehg* will be equal parallelograms; being equal, because they are formed by planes perpendicular to the same straight line, and consequently parallel; and being parallelograms, because *a*B, *dc*, two opposite sides of the same section, are formed by the meeting of one plane with two parallel planes ABFE, DCGH.

For a like reason, the figure B*ae*F is a parallelogram; so also are BF*gc*, *cdhg* and *adhe*, the other lateral faces of the solid B*adc*F*ehg*: hence that solid is a prism (Def. 4); and that prism is a right one, because the side BF is perpendicular to its base.

This being proved, if the right prism B*h* be divided by the plane BFHD into two right triangular prisms *a*B*de*F*h*, BdcF*hg*, it will remain to be shown that the oblique triangular prism ABDEFH will be equal to the right triangular prism *a*B*de*F*h*; and since those two prisms have a part ABD*he*F in common, it will only be requisite to prove that the remaining parts, namely, the solids B*a*AD*d*, F*e*EH*h*, are equal.

Now, by reason of the parallelograms ABFE, *a*BF*e*, the sides AE, *ae*, being equal to their parallel BF, are equal to each other; and taking away the common part A*e*, there remains A*a*=E*e*. In the same manner we could prove D*d*=H*h*.

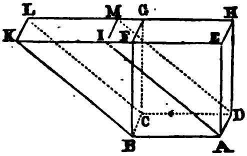
Let us now place the base F*eh* on its equal B*ad*; the point *e* coinciding with *a*, and the point *h* with *d*, the sides *e*E, *h*H will coincide with their equals *a*A, *d*D, because they are perpendicular to the same plane B*ad*. Hence the two solids in question will coincide exactly with each other, and the oblique prism BADFEH is therefore equal to the right one B*ad*F*eh*.

In the same manner might the oblique prism BDCFHG be proved equal to the right prism B*dc*F*hg*. But (B. VII, Prop. 1) the two right prisms B*ad*F*eh*, B*dc*F*hg* are equal, since they have the same altitude BF, and since their bases B*ad*, B*dc* are halves of the same parallelogram. Hence the two triangular prisms BADFEH, BDCFHG, being equal to the equal oblique prisms, are equal to each other.

*Cor.* Every triangular prism ABDHEF is half of the parallelopipedon AG described on the same solid angle A, with the same edges AB, AD, AE.

### PROPOSITION V.

**Theorem.** *Two parallelopipedons having a common base, and their upper bases in the same plane and between the same parallels, are equal to each other.*



Let the two parallelopipedons AG, AL have the common base ABCD; and let their upper bases EFGH, IKLM be in the same plane, and between the same parallels EK, HL: then will these parallelopipedons be equal.

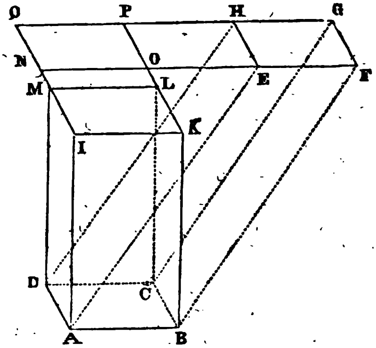
There may be three cases to this proposition, according as EI is greater, less than, or equal to EF; but the demonstration is the same for all. In the first place, then, we shall show that the triangular prism AEIDHM is equal to the triangular prism BFKCGL.

Since AE is parallel to BF, and HE to GF, the angle AEI=BFK, HEI=GFK, and HEA=GFB. Of these six angles, the first three form the solid angle E, and the last three the solid angle F; therefore, the plane angles being respectively equal and similarly arranged, the solid angles F and E must be equal. Now, if the prism AEM be laid on the prism BFL, the base AEI, being placed on the base BFK, will coincide with it, because they are equal; and since the solid angle E is equal to the solid angle F, the side EH will coincide with its equal FG; and nothing more is required to prove the coincidence of the two prisms throughout their whole extent, for (B. VII, Prop. 1) the base AEI and the edge EH determine the prism AEM, as the base BFK and the edge FG determine the prism BFL: hence these prisms are equal.

But if the prism AEM is taken away from the solid AL, there will remain the parallelopipedon AIL; and if the prism BFL is taken away from the same solid, there will remain the parallelopipedon AEG: hence these two parallelopipedons AIL, AEG are equal.

### PROPOSITION VI.

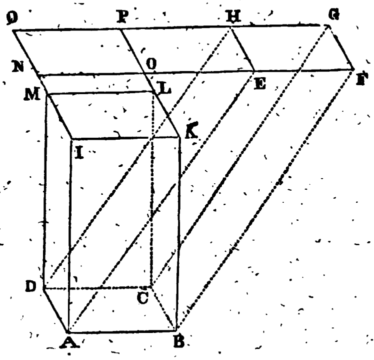
THEOREM. *Two parallelopipedons having the same base and same altitude, are equal to each other.*



Let ABCD be the common base of the two parallelopipedons AG, AL: since they have the same altitude, their upper bases EFGH, IKLM will be in the same plane. Also the sides EF and AB will be equal and parallel, as well as IK and AB; hence EF is equal and parallel to IK: for a like reason, GF is equal and parallel to LK. Let the sides EF, HG be produced, and likewise LK, IM, till, by their intersections, they form the parallelogram NOPQ: this parallelogram will evidently be equal to either of the bases EFGH, IKLM. Now if a third parallelopipedon be conceived, having ABCD for its lower base and NOPQ for its upper, this third parallelopipedon will (B. VII, Prop. v) be equal to the parallelopipedon AG; since, with the same lower base, their upper bases lie in the same plane and between the same parallels GQ, FN. For the same reason, this third parallelopipedon will also be equal to the parallelopipedon AL: hence the two parallelopipedons AG, AL, which have the same base and the same altitude, are equal.

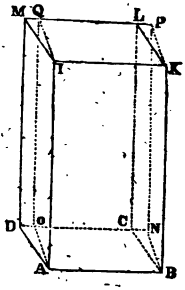
### PROPOSITION VII.

THEOREM. *Any parallelopipedon may be changed into an equal rectangular parallelopipedon having the same altitude and an equal base.*



Let AG be the parallelopipedon proposed. From the points A, B, C, D draw AI, BK, CL, DM perpendicular to the plane of the base; and we shall thus form the parallelopipedon AL equal to AG, and having its lateral faces AK, BL, &c. rectangular. Hence if the base ABCD be a rectangle, AL will be the rectangular parallelopipedon equal to AG the parallelopipedon proposed.

But if ABCD is not a rectangle, draw AO and BN perpendicular to CD, and OQ and NP perpendicular to the base; then the solid ABNOIPQ will be a rectangular parallelopipedon: for, by construction, the base ABNO and its opposite IKPQ are rectangles; so also are the lateral faces, the edges AI, OQ, &c. being perpendicular to the plane of the base; hence the solid AP is a rectangular parallelopipedon. But the two parallelopipedons AP, AL may be conceived as having the same base ABKI, and the same altitude AO : hence the parallelopipedon AG, which was at first changed into an equal parallelopipedon AL, is again changed into an equal rectangular parallelopipedon AP, having the same altitude AI, and a base ABNO equal to the base ABCD.

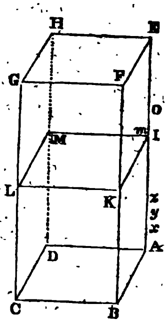


### PROPOSITION VIII.

**Theorem.** *Two rectangular parallelopipedons having the same base, are to each other as their altitudes.*

Let the parallelopipedons AG, AL have the common base ABCD; then will they be to each other as their altitudes AE, AI.

First, suppose the altitudes AE, AI to be to each other as two whole numbers; for example, as 15 is to 8. Divide AE into 15 equal parts, whereof AI will contain 8; and through $x$, $y$, $z$, &c. the points of division, draw planes pa-



rallel to the base. These planes will cut the solid AG into 15 partial parallelopipedons, all equal to each other, having equal bases and equal altitudes : equal bases, because every section MIKL, made parallel to the base ABCD of a prism, is equal to that base (B. VII, Prop. III, Cor.); and equal altitudes, because these altitudes are the same divisions $Ax$, $xy$, $yz$, &c. But of those 15 equal parallelopipedons, 8 are contained in AL : hence the solid AG is to the solid AL as 15 is to 8; or, generally, as the altitude AE is to the altitude AI.

Again, if the ratio of AE to AI cannot be expressed in numbers, it is to be shown that, notwithstanding, we shall have solid AG : solid AL : : AE : AI.

For, if this proportion is not correct, suppose we have

solid AG : solid AL : : AE : AO, greater than AI. Divide AE into equal parts, such that each shall be less than OI; there will be at least one point of division $m$ between O and I. Let P be the parallelopipedon, whose base is ABCD and altitude A$m$. Since the altitudes AE, A$m$ are to each other as two whole numbers, we shall have solid AG : P : : AE : A$m$.

But, by hypothesis, we have

solid AG : solid AL : : AE : AO; therefore

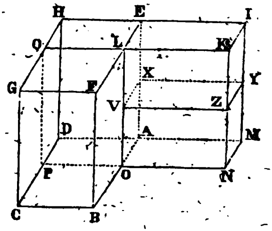
solid AL : P : : AO : A$m$.

But AO is greater than A$m$; hence, if the proportion is correct, the solid AL must be greater than P. On the contrary, however, it is less; hence the fourth term of this proportion, solid AG : solid AL : : AE : $x$, cannot possibly be a line greater than AI.

By the same mode of reasoning, it might be shown that the fourth term cannot be less than AI; therefore it is equal to AI. Hence rectangular parallelopipedons having the same base, are to each other as their altitudes.

### PROPOSITION IX.

**Theorem.** *Two rectangular parallelopipedons having the same altitude, are to each other as their bases.*



Let the two rectangular parallelopipedons AG, AK have the same altitude; then will they be to each other as their bases ABCD, AMNO.

Produce the plane ONKL till it meets the plane DCGH in PQ; we shall thus have a third parallelopipedon AQ, which may be compared with each of the parallelopipedons AG, AK. The two solids AG, AQ having the same base AEHD, are to each other as their altitudes AB, AO: In like manner, the two solids AQ, AK having the same base AOLE, are to each other as their altitudes AD, AM. Hence we have the two proportions

tions sol. AG : sol. AQ : : AB : AO;
sol. AQ : sol. AK : : AD : AM.

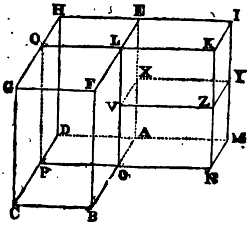
Multiply together the corresponding terms of these proportions, omitting in the result the common multiplier sol. AQ: we shall have

sol. AG : sol. AK : : AB × AD : AO × AM.

But AB × AD represents the base ABCD, and AO × AM represents the base AMNO: hence two rectangular parallelopipedons of the same altitude are to each other as their bases.

### PROPOSITION X.

THEOREM. *Any two rectangular parallelopipedons are to each other as the products of their bases by their altitudes; that is to say, as the products of their three dimensions.*



For, having placed the two solids AG, AZ so that their surfaces have the common angle BAE, produce the interior planes necessary for completing the third parallelopipedon AK, having the same altitude with the parallelopipedon AG. By the last proposition, we shall have

sol. AG : sol. AK : : ABCD : AMNO.

But the two parallelopipedons AK, AZ having the same base AMNO, are to each other as their altitudes AE, AX: hence we have

sol. AK : sol. AZ : : AE : AX.

Multiply together the corresponding terms of these proportions, omitting in the result the common multiplier sol. AK: we shall have

sol. AG : sol. AZ : : ABCD × AE : AMNO × AX.

Instead of the bases ABCD and AMNO, put AB × AD and AO × AM: it will give

sol. AG : sol. AZ : : AB × AD × AE : AO × AM × AX.

Hence any two rectangular parallelopipedons are to each other, etc.

*Scholium.* We are consequently authorised to assume, as the measure of a rectangular parallelopipedon, the product of its base by its altitude; in other words, the product of its three dimensions.

In order to comprehend the nature of this measurement, it is necessary to reflect, that by the product of two or more lines is always meant the product of the numbers which represent them; those numbers themselves being determined by their linear unit, which may be assumed at pleasure. Upon this principle, the product of the three dimensions of a parallelopipedon is a number, which signifies nothing of itself, and would be different if a different linear unit had been assumed; but if the three dimensions of another parallelopipedon are valued according to the same linear unit, and multiplied together in the same manner, the two products will be to each other as the solids, and will serve to express their relative magnitude.

The magnitude of a solid, its volume or extent, form what is called its *solidity;* and this word is exclusively employed to designate the measure of a solid : thus we say the solidity of a rectangular parallelopipedon is equal to the product of its base by its altitude, or to the product of its three dimensions.

As the cube has all its three dimensions equal, if the side is 1, the solidity will be $1\times1\times1=1$; if the side is 2, the solidity will be $2\times2\times2=8$; if the side is 3, the solidity will be $3\times3\times3=27$; and so on : hence, if the sides of a series of cubes are to each other as the numbers 1, 2, 3, &c., the cubes themselves (or their solidities) will be as the numbers 1, 8, 27, &c. Hence it is, that in arithmetic, the *cube* of a number is the name given to the product which results from three factors, each equal to this number.

If it were proposed to find a cube double of a given cube, the side of the required cube would have to be to that of the given one, as the cube root of 2 is to unity. Now, by a geometrical construction, it is easy to find the square root of 2; but the cube root of it cannot be so found, at least not by the simple operations of elementary geometry, which consist in employing nothing but straight lines, two points of which are known, and circles whose centres and radii are determined.

Owing to this difficulty, the problem of *the duplication of the cube* became celebrated among the ancient geometers, as well as that of *the trisection of an angle*, which is nearly of the same species. The solutions of which such problems are susceptible, have, however, long since been discovered; and though less simple than the constructions of elementary geometry, they are not, on that account, less rigorous or less satisfactory.

### PROPOSITION XI.

THEOREM. *The solidity of a parallelopipedon, and generally of any prism, is equal to the product of its base by its altitude.*

For, in the first place, any parallelopipedon (B. VII, Prop. VII) is equal to a rectangular parallelopipedon having the same altitude and an equal base. Now the solidity of the latter is equal to its base multiplied by its height:

hence the solidity of the former is, in like manner, equal to the product of its base by its altitude.

In the second place, and for a like reason, any rectangular prism is half of the parallelopipedon so constructed as to have the same altitude and a double base; but the solidity of the latter is equal to its base multiplied by its altitude : hence that of a triangular prism is also equal to the product of its base multiplied into its altitude.

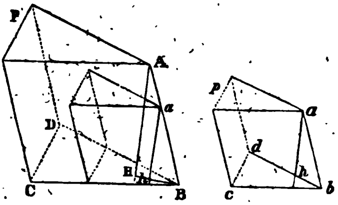
In the third place, any prism may be divided into as many triangular prisms of the same altitude, as there are triangles capable of being formed in the polygon which constitutes its base : but the solidity of each triangular prism is equal to its base multiplied by its altitude; and since the altitude is the same for all, it follows that the sum of all the partial prisms must be equal to the sum of all the partial triangles which constitute their bases, multiplied by the common altitude.

Hence the solidity of any polygonal prism is equal to the product of its base by its altitude.

*Cor.* Comparing two prisms which have the same altitude, the products of their bases by their altitudes will be as the bases simply : hence *two prisms of the same altitude are to each other as their bases.* For a like reason, *two prisms of the same base are to each other as their altitudes.*

## PROPOSITION XII.

THEOREM. *Similar prisms are to one another as the cube of their homologous sides.*



Let P and *p* be two prisms, of which BC, *bc* are homologous sides: the prism P is to the prism *p*, as the cube of BC to the cube of *bc*.

From A and *a*, homologous angles of the two prisms, draw AH, *ah* perpendicular to the bases BCD, *bcd*: join BH; take B*a* = *ba*, and in the plane BHA draw *ah* perpendicular to BH: then *ah* shall be perpendicular to the plane CBD, and equal to *ah* the altitude of the other prism; for if the solid angles B and *b* were applied the one to the other, the planes which contain them, and consequently the perpendiculars *ah*, *ah* would coincide.

Now because of the similar triangles ABH, *abh*, and the similar figures AC, *ac*, we have

AH : *ah* :: AB : *ab* :: BC : *bc*;

and because of these similar bases, the

base BCD : base *bcd* :: $BC^2 : bc^2$. [B. IV, Prop. XVI.]

Taking the product of the corresponding terms of these proportions, we have

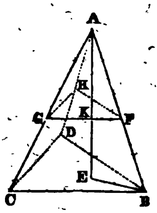
AH × base BCD : *ah* × base *bcd* :: $BC^3 : bc^3$.

But AH × base BCD expresses the solidity of the prism P, and $ah \times$ base $bcd$ expresses the solidity of the other prism $p$; therefore

$$\text{prism P} : \text{prism } p :: \text{BC}^2 : bc^2.$$

### PROPOSITION XIII.

THEOREM. *The section formed by cutting a triangular pyramid by a plane parallel to the base, is similar to the base.*



Let the triangular pyramid A – BCD be cut by a plane parallel to the base, forming the section FGH; then will this section be similar to the base BCD.

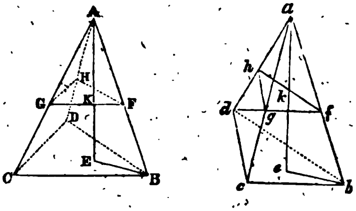
For because the parallel planes BCD, FGH are cut by a third plane ABC, the sections FG, BC are parallel (B VI, Prop. x). In like manner, it appears that FH is parallel to BD; therefore the angle HFG is equal to the angle DBC (B. VI, Prop. xiii); and because the triangle ABC is similar to the triangle AFG, and the triangle ABD is similar to the triangle AFH, we have

BC : BA : : FG : FA, and
BA : BD : : FA : FH; therefore
BC : BD : : FG : FH.

Now the angle DBC has been shown to be equal to the angle HFG; therefore the triangles DBC, HFG are equiangular (B. IV, Prop. x).

### PROPOSITION XIV.

THEOREM. *If two triangular pyramids, which have equal bases and equal altitudes, be cut by planes which are parallel to the bases, and at equal distances from them; those sections will be equal.*



Let the two triangular pyramids ABCD, *abcd*, which have equal bases and altitudes, be cut by planes parallel to the bases and at equal distances, forming the sections FGH, *fgh*; then will these sections be equal.

Draw AKE, *ake* perpendicular to the bases BCD, *bcd*, meeting the cutting planes in K and *k*; then, because of the parallel planes, we have (B. VI, Prop. xv)

AE : AK : : AB : AF, and *ae* : *ak* : : *ab* : *af*;

but, by hypothesis, AE = *ae*, and AK = *ak*; therefore

AB : AF : : *ab* : *af*.

Again, because of similar triangles,

AB : AF : : BC : FG, and *ab* : *af* : : *bc* : *fg*;

and, hence, $BC^2 : FG^2 :: bc^2 : fg^2$;

but, because of the similar triangles BDC, FGH,

$BC^2 : FG^2 ::$ triangle BDC : triangle FHG.

And, in like manner,

$bc^2 : fg^2 ::$ triangle *bdc* : triangle *fgh*; therefore

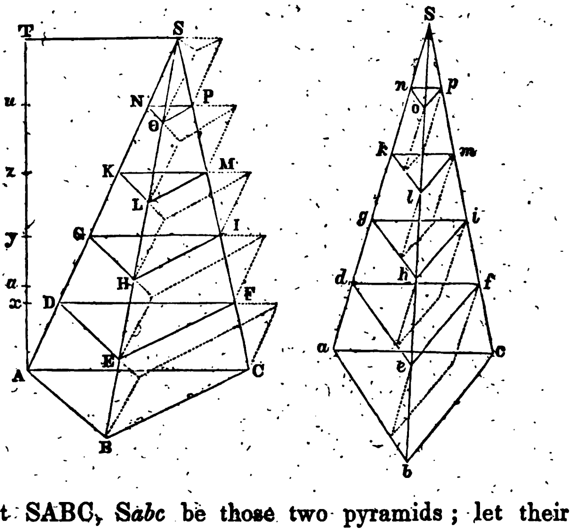
trian. BCD : trian. FGH : : trian. *bcd* : trian. *fgh*.

Now trian. BCD = trian. *bcd* (by hypothesis); therefore the triangle FHG is equal to the triangle *fhg*.

*Scholium.* It is easy to see, that what is proved in this and the preceding proposition, is also true of polygonal pyramids.

### PROPOSITION XV.

THEOREM. *Two triangular pyramids having equivalent bases and equal altitudes, are equivalent, or equal in solidity.*



Let SABC, S*abc* be those two pyramids; let their equivalent bases ABC, *abc* be situated in the same plane, and let AT be their common altitude. If they are not equivalent, let S*abc* be the smaller; and suppose A*a* to be the altitude of a prism, which, having ABC for its base, is equal to their difference.

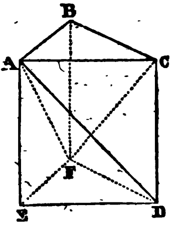
Divide the altitude AT into equal parts A*x*, *xy*, *yz*, &c. each less than A*a*, and let *k* be one of those parts; through the points of division, pass planes parallel to the plane of the bases: the corresponding sections formed by these planes in the two pyramids will be respectively equivalent (B. VII, Prop. xiv), namely, DEF to *def*, GHI to *ghi*, &c.

This being granted, upon the triangles ABC, DEF, GHI, &c. taken as bases, construct exterior prisms having for edges the parts AD, DG, GK, &c. of the edge SA. In like manner, on the bases *def*, *ghi*, *klm*, &c. in the second pyramid, construct interior prisms, having for edges the corresponding parts of S*a*. It is plain that the sum of all the exterior prisms of the pyramid SABC will be greater than this pyramid; and, also, that the sum of all the interior prisms of the pyramid S*abc* will be less than this. Hence the difference between the sum of all the exterior prisms and the sum of all the interior ones, must be greater than the difference between the two pyramids themselves.

Now, beginning with the bases ABC, *abc*, the second exterior prism DEFG is equivalent to the first interior prism *defa*, because they have the same altitude *k*, and their bases DEF, *def* are equivalent: for like reasons, the third exterior prism GHIK, and the second interior prism *ghid*, are equivalent; the fourth exterior, and the third interior; and so on, to the last in each series. Hence all the exterior prisms of the pyramid SABC, excepting the first prism DABC, have equivalent corresponding ones in the interior prisms of the pyramid S*abc*: hence the prism DABC is the difference between the sum of all the exterior prisms of the pyramid SABC, and the sum of all the interior prisms of the pyramid $Sabc$. But the difference between these two sets of prisms has already been proved to be greater than that of the two pyramids, which latter difference we supposed to be equal to the prism $aABC$: hence the prism DABC must be greater than the prism $aABC$; but in reality it is less, for they have the same base ABC, and the altitude $Ax$ of the first is less than $Aa$ the altitude of the second. Hence the supposed inequality between the two pyramids cannot exist: hence the two pyramids SABC, $Sabc$, having equal altitudes and equivalent bases, are themselves equivalent.

### PROPOSITION XVI.

**Theorem.** *Every triangular pyramid is the third of the triangular prism having the same base and altitude.*



Let FABC be a triangular pyramid, ABCDEF a triangular prism of the same base and altitude: the pyramid will be equal to one-third of the prism.

Conceive the pyramid FABC to be cut off from the prism by a section made along the plane FAC, and there will remain the solid FACDE, which may be considered as a quadrangular pyramid whose vertex is F, and base the parallelogram ACDE. Draw the diagonal AD, and extend the plane FAD, which will cut the quadrangular pyramid into two triangular ones FACD, FADE. These two triangular pyramids have for their common altitude the perpendicular drawn from F to the plane ACDE; they have equal bases, the triangles ACD, ADE being halves of the same parallelogram : hence the two pyramids FACD, FADE are equal. But the pyramid FADE and the pyramid FABC have equal bases ABC, DEF; they have also the same altitude, namely, the distance of the parallel planes ABC, DEF : hence the two pyramids are equal. Now the pyramid FADE has already been proved equal to FACD; hence the three pyramids FABC, FADE, FACD, which compose the prism ABCD, are all equal. Hence the pyramid FABC is the third part of the prism ABCD, which has the same base and the same altitude.

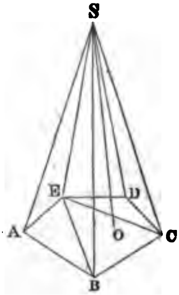
*Cor.* The solidity of a triangular pyramid is equal to a third part of the product of its base by its altitude.

### PROPOSITION XVII.

THEOREM. *Any pyramid is measured by a third part of the product of its base by its altitude.*

Let S – ABCDE be a pyramid, having the altitude SO; then will it be measured by the base ABCDE into one-third of the altitude SO.

For, extending the planes SEB, SEC through the diagonals EB, EC, the polygonal pyramid SABCDE will be divided into several triangular pyramids, all having the same altitude SO. But (B. VII, Prop. xvi) each of these pyramids is measured



by multiplying its base ABE, BCE or CDE by the third part of its altitude SO : hence the sum of these triangular pyramids, or the polygonal pyramid SABCDE, will be measured by the sum of the triangles ABE, BCE, CDE, or the polygon ABCDE, multiplied by $\frac{1}{3}$ SO. Hence every pyramid is measured by a third part of the product of its base by its altitude.

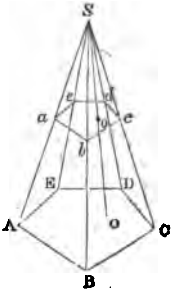
*Cor.* 1. Every pyramid is the third part of the prism which has the same base and the same altitude.

*Cor.* 2. Two prisms having the same altitude, are to each other as their bases.

*Scholium.* The solidity of any polyedral body may be computed, by dividing the body into pyramids; and this division may be accomplished in various ways. One of the simplest is to make all the planes of division pass through the vertex of one solid angle; in that case, there will be formed as many partial pyramids as the polyedron has faces, *minus* those faces which form the solid angle whence the planes of division proceed.

### PROPOSITION XVIII.

THEOREM. *Two similar pyramids are to each other as the cubes of their homologous sides.*



For, two pyramids being similar, the smaller may be placed within the greater, so that the solid angle S shall be common to both. In that position the bases ABCDE, *abcde* will be parallel; because, since the homologous faces are similar, the angle S*ab* is equal to SAB, and S*bc* to SBC : hence the plane ABC is parallel to the plane *abc*. This granted, let SO be the perpendicular drawn from the vertex S to the plane ABC, and *o* the point where this perpendicular meets the plane *abc* : from what has already been shown (B. VI, Prop. xv), we shall have

$$\mathrm{SO} : \mathrm{S}o :: \mathrm{SA} : \mathrm{S}a :: \mathrm{AB} : ab\text{; and,}$$

consequently, $\frac{1}{3}\,\mathrm{SO} : \frac{1}{3}\,\mathrm{S}o :: \mathrm{AB} : ab.$

But the bases ABCDE, *abcde* being similar figures, we have $\mathrm{ABCDE} : abcde :: \mathrm{AB}^2 : ab^2.$

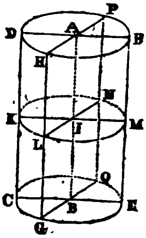
Multiply the corresponding terms of these two proportions; there results the proportion,

$$\mathrm{ABCDE} \times \tfrac{1}{3}\,\mathrm{SO} : abcde \times \tfrac{1}{3}\,\mathrm{S}o :: \mathrm{AB}^3 : ab^3.$$

Now ABCDE $\times \frac{1}{3}$ SO is the solidity of the pyramid SABCDE, and *abcde* $\times \frac{1}{3}$ S*o* is that of the pyramid S*abcde* (B. VII, Prop. XVII) : hence two similar pyramids are to each other as the cubes of their homologous sides.

# BOOK EIGHTH.

## DEFINITIONS.



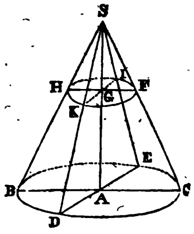
1. A *cylinder* is a solid, which may be produced by the revolution of a rectangle ABCD, conceived to turn about the immovable side AB.

In this rotation, the sides AD, BC, continuing always perpendicular to AB, describe equal circular planes DHP, CGQ, which are called *the bases of the cylinder;* the side CD at the same describing *the convex surface.*

The immovable line AB is called *the axis of the cylinder.*

Every section KLM made in the cylinder, at right angles to the axis, is a circle equal to either of the bases; for, while the rectangle ABCD revolves about AB, the line KI, perpendicular to AB, describes a circular plane, equal to the base, which is a section made perpendicular to the axis at the point I.

Every section PQGH passing through the axis, is a rectangle, and is double of the generating rectangle ABCD.



2. A *cone* is a solid, which may be produced by the revolution of a right-angled triangle SAB, conceived to turn about the immovable side SA.

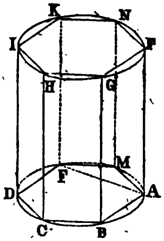
In this rotation, the side AB describes a circular plane BDCE, named *the base of the cone;* and the hypothenuse SB, its *convex surface.*

The point S is named *the vertex of the cone;* SA, its *axis* or *altitude.*

Every section HKFI formed at right angles to the axis, is a circle. Every section SDE passing through the axis, is an isosceles triangle double of the generating triangle SAB.

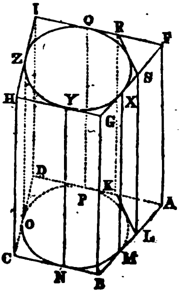
3. If, from the cone SCDB, the cone SFKH be cut off by a section parallel to the base, the remaining solid CBHF is called *a truncated cone*, or *the frustum of a cone.* We may conceive it to be described by the revolution of a trapezium ABHG, whose angles A and C are right, about the side AG. The immovable line AG is called *the axis* or *altitude of the frustum;* the circles BDC, HFK are its *bases*, and BH is its *side.*

4. Two cylinders, or two cones, are *similar*, when their axes are to each other as the diameters of their bases.



5. If, in the circle ACD which forms the base of a cylinder, a polygon ABCDE is inscribed, a right prism, constructed on this base ABCDE, and equal in altitude to the cylinder, is said to be *inscribed in the cylinder*, or the cylinder to be *circumscribed about the prism.*

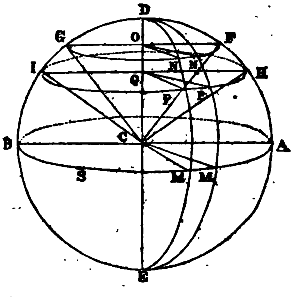
The edges AF, BG, CH, &c. of the prism, being perpendicular to the plane of the base, are evidently included in the convex surface of the cylinder: hence the prism and the cylinder touch one another along these edges.



6. In like manner, if ABCD is a polygon circumscribed about the base of a cylinder, a right prism, constructed on this base ABCD, and equal in altitude to the cylinder, is said to be *circumscribed about the cylinder*, or the cylinder to be *inscribed in the prism.*

Let M, N, &c. be the points of contact in the sides AB, BC, &c.; and through the points M, N, &c. let MX, NY, &c. be drawn perpendicular to the plane of the base: those perpendiculars will evidently lie both in the surface of the cylinder, and in that of the circumscribed prism; hence they will be their lines of contact.

7. A *sphere* is a solid terminated by a curve surface, all the points of which are equally distant from a point within, called *the centre.*



The sphere may be conceived to be generated by the revolution of a semicircle DAE about its diameter DE; for the surface described in this movement, by the curve DAE, will have all its points equally distant from the centre C.

8. *The radius of a sphere*, is a straight line drawn from the centre to any point in the surface; *the diameter*, or *axis*, is a line passing through this centre, and terminated on both sides by the surface.

All the radii of a sphere are equal: all the diameters are equal, and double of the radius.

9. *A great circle of the sphere,* is a section which passes through the centre; *a small circle,* one which does not pass through it.

10. *A plane is a tangent* to a sphere, when their surfaces have but one point in common.

11. *The pole of a circle* of a sphere, is a point in the surface equally distant from all the points in the circumference of this circle.

12. *A spherical triangle* is a portion of the surface of a sphere, bounded by three arcs of great circles.

Those arcs, named *the sides of the triangle,* are always supposed to be each less than a semicircumference; the angles, which their planes form with each other, are *the angles of the triangle.*

13. A spherical triangle takes the name of *right-angled, isosceles, equilateral,* in the same cases as a rectilineal triangle.

14. *A spherical polygon* is a portion of the surface of a sphere, terminated by several arcs of great circles.

15. *A lune* is that portion of the surface of a sphere, which is included between two great semicircles meeting in a common diameter.

16. *A spherical wedge,* or *ungula,* is that portion of the solid sphere, which is included between the same great semicircles, and has the lune for its base.

17. *A spherical pyramid* is a portion of the solid sphere, included between the planes of a solid angle whose vertex is the centre; *the base* of the pyramid, is the spherical polygon intercepted by the same planes,

18. *A zone* is the portion of the surface of the sphere, included between two parallel planes, which form its *bases*. One of these planes may be a tangent to the sphere; in which case, the zone has only a single base.

19. *A spherical segment* is the portion of the solid sphere, included between two parallel planes which form its bases. One of those planes may be a tangent to the sphere; in which case, the segment has only a single base.

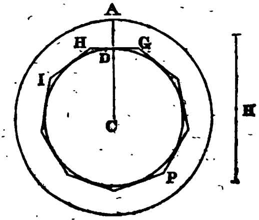
20. *The altitude of a zone*, or of *a segment*, is the distance of the two parallel planes, which form the bases of the zone or segment.

21. While the semicircle DAE (Def. 7), revolving round its diameter DE, describes the sphere; any circular sector, as DCF or FCH, describes a solid, which is named *a spherical sector*.

*Note.* The cylinder, the cone, and the sphere, are *the three round bodies* treated of in the elements of geometry.

### PROPOSITION I.

THEOREM. *The solidity of a cylinder is equal to the product of its base by its altitude.*



Let CA be a radius of the given cylinder's base; H the altitude. Let surf. CA represent the area of the circle whose radius is CA: we are to show that the solidity of the cylinder is surf. CA × H. For, if surf. CA × H is not the measure of the given cylinder, it must be the measure of a greater cylinder, or of a smaller one. Suppose it first to be the measure of a smaller one; of a cylinder, for example, which has CD for the radius of its base, H being the altitude.

About the circle whose radius is CD, circumscribe a regular polygon GHIP, the sides of which shall not meet the circumference whose radius is CA. Imagine a right prism, having the regular polygon GHIP for its base, and H for its altitude; this prism will be circumscribed about the cylinder, whose base has CD for its radius. Now (B. VII, Prop. XI) the solidity of the prism is equal to its base GHIP multiplied by the altitude H: the base GHIP is less than the circle whose radius is CA; hence the solidity of the prism is less than surf. CA × H. But, by hypothesis, surf. CA × H is the solidity of the cylinder inscribed in the prism; hence the prism must be less than the cylinder, whereas in reality it is greater, because it contains the cylinder: hence it is impossible that surf. CA × H can be the measure of the cylinder whose base has CD for its radius, H being the altitude; or, in more general terms, *the product of the base by the altitude of a cylinder, cannot measure a less cylinder.*

We must now prove that the same product cannot measure a greater cylinder. To avoid the necessity of changing our figure, let CD be a radius of the given cylinder's base; and, if possible, let surf. CD × H be the measure of a greater cylinder, for example, of the cylinder whose base has CA for its radius, H being the altitude.

The same construction being performed as in the first case, the prism, circumscribed about the given cylinder, will have GHIP × H for its measure : the area GHIP is greater than surf. CD; hence the solidity of this prism is greater than surf. CD × H; hence the prism must be greater than the cylinder having the same altitude, and surf. CA for its base. But, on the contrary, the prism is less than the cylinder, being contained in it; therefore *the base of a cylinder multiplied by its altitude, cannot be the measure of a greater cylinder.*

Hence, finally, the solidity of a cylinder is equal to the product of its base by its altitude.

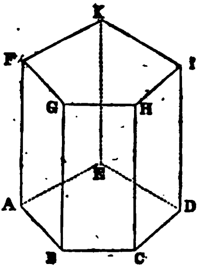
*Cor.* 1. Cylinders of the same altitude are to each other as their bases; and cylinders of the same base are to each other as their altitudes.

*Cor.* 2. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases. For the bases are as the squares of their diameters; and the cylinders being similar, the diameters of their bases (Def. 4) are to each other as the altitudes : hence the bases are as the squares of the altitudes; hence the bases multiplied by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

*Schol.* Let R be the radius of a cylinder's base, and H the altitude : the surface of the base (B. V, Prop. xiv, Schol.) will be $\pi R^2$; and the solidity of the cylinder will be $\pi R^2 \times H$, or $\pi R^2 H$, where $\pi = 3 \cdot 141592$, &c.

### PROPOSITION II.

**Theorem.** *The convex surface of a right prism is equal to the perimeter of it's base multiplied by its altitude.*

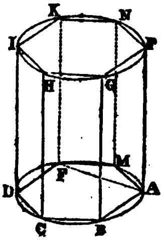


For this surface is equal to the sum of the rectangles AFGB, BGHC, CHID, &c. which compose it. Now the altitudes AF, BG, CH, &c. of those rectangles, are equal to the altitude of the prism; their bases AB, BC, CD, &c. taken together, make up the perimeter of the prism's base : hence the sum of these rectangles, or the convex surface of the prism, is equal to the perimeter of its base multiplied by its altitude.

*Cor.* If two right prisms have the same altitude, their convex surfaces will be to each other as the perimeters of their bases.

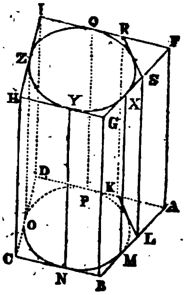
### PROPOSITION III.

**Theorem.** *The convex surface of a cylinder is greater than the convex surface of any inscribed prism, and less than the convex surface of any circumscribed prism.*



For the concave surface of the cylinder and that of the prism may be considered as having the same length, since every section made in either parallel to AF is equal to AF ; and if these surfaces be cut, in order to obtain the breadths of them, by planes parallel to the base, or perpendicular to the edge

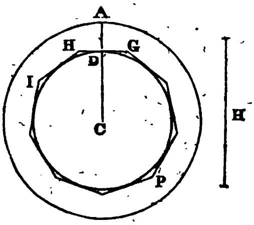
AF, the one section will be equal to the circumference of the base, the other to the contour of the polygon ABCDE, which is less than that circumference: hence, with an equal length, the cylindrical surface is broader than the prismatic surface; hence the former is greater than the latter.



By a similar demonstration, the convex surface of the cylinder might be shown to be less than that of any circumscribed prism BCDKLKH.

### PROPOSITION IV.

THEOREM. *The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude.*



Let CA be the radius of the given cylinder's base, and H its altitude; the circumference whose radius is CA being represented by circ. CA, we are to show that circ. CA × H will be the convex surface of the cylinder. For, if this proposition be not true, then circ. CA × H must be the convex surface of a greater cylinder, or of a less one. Suppose it first to be the surface of a less cylinder; of the cylinder, for example, the radius of whose base is CD, and whose altitude is H.

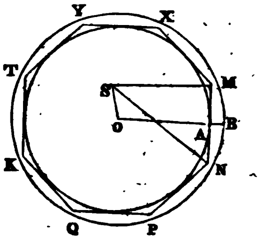
About the circle whose radius is CD, circumscribe a regular polygon GHIP, the sides of which shall not meet the circle whose radius is CA. Conceive a right prism having H for its altitude, and the polygon GHIP for its base. The convex surface of this prism will be equal (B. VIII, Prop. II) to the contour of the polygon GHIP multiplied by the altitude H : this contour is less than the circumference whose radius is CA; hence the convex surface of the prism is less than circ. CA × H. But, by hypothesis, circ. CA × H is the convex surface of the cylinder whose base has CD for its radius, which cylinder is inscribed in the prism; hence the convex surface of the prism must be less than that of the inscribed cylinder; but, by hypothesis (B. VIII, Prop. III), it is greater : hence, in the first place, *the circumference of a cylinder's base, multiplied by its altitude, cannot be the measure of a smaller cylinder.*

Neither can this product be the measure of a greater cylinder. For, retaining the present figure, let CD be the radius of the given cylinder's base; and, if possible, let circ. CD × H be the convex surface of a cylinder, which, with the same altitude, has for its base a greater circle, the circle, for instance, whose radius is CA. The same construction being performed as above, the convex surface of the prism will again be equal to the contour of the polygon GHIP multiplied by the altitude H; but this contour is greater than circ. CD; therefore the surface of the prism must be greater than circ. CD × H, which, by hypothesis, is the surface of the cylinder having the same altitude, and CA for the radius of its base : hence the surface of the prism must be greater than that of the cylinder. Now if this prism were inscribed in the cylinder, its surface (B. VIII, Prop. III) would be less than the cylinder's; much more, then, is it less when the prism does not reach so far as to touch the cylinder. Consequently, also, in the second place, *the circumference of a cylinder's base, multiplied by the altitude, cannot measure the surface of a greater cylinder.*

The product in question, being, therefore, neither the measure of the convex surface of a less nor greater cylinder, must be the measure of the cylinder itself.

### PROPOSITION V.

THEOREM. *The solidity of a cone is equal to the product of its base by the third of its altitude.*



Let SO be the altitude of the given cone, AO the radius of its base; the surface of the base being designated by surf. AO, it is to be demonstrated that surf. AO × $\frac{1}{3}$ SO is equal to the solidity of the cone.

Suppose, first, that surf. AO × $\frac{1}{3}$ SO is the solidity of a greater cone; for example, of the cone whose altitude is also SO, but whose base has OB greater than AO for its radius.

About the circle whose radius is AO, circumscribe a regular polygon MNPT, so as not to meet the circumference whose radius is OB: imagine a pyramid having this polygon for its base, and the point S for its vertex. The solidity of this pyramid (B. VII, Prop. XVII) is equal to the area of the polygon MNPT multiplied by a third of the altitude SO; but the polygon is greater than the inscribed circle represented by surf. AO : hence the pyramid is greater than surf. AO × $\frac{1}{3}$ SO, which, by hypothesis, measures the cone having S for its vertex and OB for the radius of its base; whereas, in reality, the pyramid is less than the cone, being contained in it. Hence, first, the base of a cone multiplied by a third of its altitude cannot be the measure of a greater cone.

Neither can this same product be the measure of a smaller cone. For now let OB be the radius of the given cone's base; and, if possible, let surf. OB × $\frac{1}{3}$ SO be the solidity of the cone having SO for its altitude, and for its base the circle whose radius is AO. The same construction being made, the pyramid SMNPT will have for its measure the area MNPT multiplied by $\frac{1}{3}$ SO; but the area MNPT is less than surf. OB : hence the measure of the pyramid must be less than surf. OB × $\frac{1}{3}$ SO, and consequently it must be less than the cone whose altitude is SO, and whose base has AO for its radius; but, on the contrary, the pyramid is greater than the cone, because the cone is contained in it. Hence, in the second place, the base of a cone multiplied by a third of its altitude cannot be the measure of a smaller one.

Consequently the solidity of a cone is equal to the product of its base by the third of its altitude.

*Cor.* A cone is the third of a cylinder having the same base and the same altitude. Whence it follows:

1. That cones of equal altitudes are to each other as their bases;

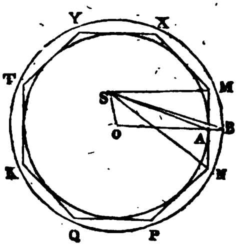
2. That cones of equal bases are to each other as their altitudes;

3. That similar cones are as the cubes of the diameters of their bases, or as the cubes of their altitudes.

*Schol.* Let R be the radius of a cone's base, and H its altitude; the solidity of the cone will be $\pi R^2 \times \frac{1}{3} H$, or $\frac{1}{3} \pi R^2 H$.

### PROPOSITION VI.

THEOREM. *The convex surface of a cone is equal to the circumference of its base multiplied by half its side.*



Let AO be a radius of the base of the given cone, S its vertex, and SA its side; the surface will be circ. AO × $\frac{1}{2}$ SA. For, if possible, let AO × $\frac{1}{2}$ SO be the surface of a cone having S for its vertex, and for its base a circle whose radius OB is greater than AO.

About the smaller circle describe a regular polygon MNPT, the sides of which may not meet the circle whose radius is OB; and let SMNPT be the regular pyramid, having this polygon for its base, and the point S for its vertex. The triangle SMN, one of those which compose the convex surface of the pyramid, has for measure its base MN multiplied by half its altitude SA, or half the side of the given cone; and since this altitude is the same in all the other triangles SNP, SPQ, &c., the convex surface of the pyramid must be equal to the perimeter MNPTM multiplied by $\frac{1}{2}$ SA. But the contour MNPTM is greater than circ. AO ; hence the convex surface of the pyramid is greater than circ. AO × $\frac{1}{2}$ SA, and consequently greater than the convex surface of the cone having the same vertex S, and the circle whose radius is OB for its base. But the surface of this cone is greater than that of the pyramid ; because, if two such pyramids are adjusted to each other base to base, and two such cones base to base, the surface of the double cone will envelope on all sides that of the double pyramid, and therefore be greater than it, as is evident ; hence the surface of the cone is greater than that of the pyramid, whereas by the hypothesis it is less. Hence, in the first place, the circumference of the cone's base multiplied by half the side cannot measure the surface of a greater cone.

Neither can it measure the surface of a smaller cone : for, let BO be the radius of the base of the given cone ; and, if possible, let circ. BO × $\frac{1}{2}$ SB be the surface of a cone having S for its vertex, and AO less than OB for the radius of its base.

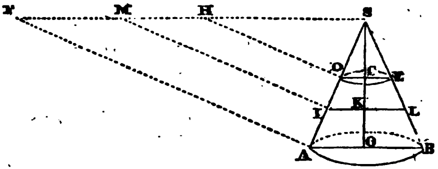
The same construction being made as above, the surface of the pyramid SMNPT will still be equal to the perimeter MNPT × $\frac{1}{2}$ SA. Now this perimeter MNPT is less than circ. OB ; likewise SA is less than SB ; consequently, for a double reason, the convex surface of the pyramid is less than circ. OB × $\frac{1}{2}$ SB, which, by hypothesis, is the surface of the cone having SA for the radius of its base : hence the surface of the pyramid must be less than that of the inscribed cone ; but it is obviously greater ; for, adjusting two such pyramids to each other base to base, and two such cones base to base, the surface of the double pyramid will envelope that of the double cone, and will be greater than it. Hence, in the second place, the circumference of the base of the given cone multiplied by half the side cannot be the measure of the surface of a smaller cone.

Therefore, finally, the convex surface of a cone is equal to the circumference of its base multiplied by half its side.

*Schol.* Let L be the side of a cone, R the radius of its base : the circumference of this base will be $2\pi R$; and the surface of the cone will be $2\pi R \times \frac{1}{2} L$, or $\pi RL$.

### PROPOSITION VII.

THEOREM. *The convex surface of a truncated cone is equal to its side multiplied by half the sum of the circumferences of its two bases.*



In the plane SAB which passes through the axis SO, draw the line AF perpendicular to SA, and equal to the circumference having AO for its radius; join SF, and draw DH parallel to AF.

From the similar triangles SAO, SDC, we have

$$AO : DC :: SA : SD;$$

and by the similar triangles SAF, SDH,

AF : DH : : SA : SD ;

hence AF : DH : : AO : DC, or (B. V, Prop. XIII, Cor. 1) as circ. AO is to circ. DC. But, by construction, AF = circ. AO ; hence DH = circ. DC. Hence the triangle SAF, measured by AF × ½ SA, is equal to the surface of the cone SAB which is measured by........ circ. AO × ½ SA. For a like reason, the triangle SDH is equal to the surface of the cone SDE. Therefore the surface of the truncated cone ADEB is equal to that of the trapezium ADHF ; but the latter (B. II, Prop. IV) is measured by $AD \times \left(\frac{AF + DH}{2}\right)$. Hence the surface of the truncated cone ADEB is equal to its side AD multiplied by half the sum of the circumferences of its two bases.

*Scholium.* If a line AD, lying wholly on one side of the line OC, and in the same plane, make a revolution around OC, the surface described by AD will have for its measure $AD \times \left(\frac{\text{circ.}AO + \text{circ.}DC}{2}\right)$, or AD × circ. IK; the lines AO, DC, IK being perpendiculars, drawn from the extremities and from the middle of the axis OC.

For, if AD and OC are produced till they meet in S, the surface described by AD is evidently that of a truncated cone having AO and DC for the radii of its bases, the vertex of the whole cone being S. Hence this surface will be measured as we have said.

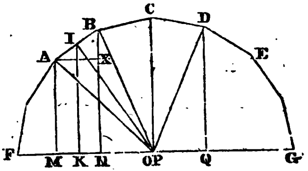
This measure will always hold good, even when the point D falls on S, and thus forms a whole cone; and also when the line AD is parallel to the axis, and thus forms a cylinder. In the first case, DC would be nothing; in the second, DC would be equal to AO and to IK.

But if the line AD is not in the same plane with OC, the surface described will be *a hyperboloid of one sheet*, whose properties cannot be fully explained without a knowledge of conic sections.

*Cor.* 1. Through I the middle point of AD, draw IKL parallel to AB, and IM parallel to AF: it may be shown, as above, that $IM = \text{circ.}\ IK$; but the

$$\text{trapezium } ADHF = AD \times IM = AD \times \text{circ. } IK.$$

Hence it may also be asserted, that *the surface of a truncated cone is equal to its side multiplied by the circumference of a section at equal distances from the two bases.*



*Cor.* 2. The point I being the middle of AB, and IK a perpendicular drawn from the point I to the axis, the surface described by AB, by the last proposition, will have for its measure $AB \times \text{circ. } IK$. Draw AX parallel to the axis; the triangles ABX, OIK will have their sides perpendicular, each to each, namely, OI to AB, IK to AX, and OK to BX; hence these triangles are similar, and give the proportion $AB : AX$ or $MN :: OI : IK$, or as circ. OI to circ. IK; hence $AB \times \text{circ. } IK = MN \times \text{circ. } OI$. Whence it is plain that the surface described by the partial polygon ABCD is measured by

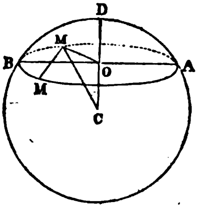
$$(MN + NP + PQ) \times \text{circ. } OI, \text{ or by } MQ \times \text{circ. } OI;$$

hence it is equal to the altitude multipled by the circumference of the inscribed circle.

*Cor.* 3. If the whole polygon has an even number of sides, and if the axis FG passes through two opposite vertices F and G, the whole surface described by the revolution of the half polygon FACG will be equal to its axis FG multipled by the circumference of the inscribed circle. This axis FG will, at the same time, be the diameter of the circumscribed circle.

## PROPOSITION VIII.

THEOREM. *Every section of a sphere, made by a plane, is a circle.*



Let AMB be the section, made by a plane in the sphere whose centre is C. From the point C, draw CO perpendicularly to the plane AMB; and draw lines CM, CM to different points of the curve AMB, which terminates the section.

The oblique lines CM, CM, CB being equal, being radii of the sphere, they are equally distant from the perpendicular CO (B. VI, Prop. v); hence all the lines OM, MO, OB are equal : hence the section AMB is a circle, whose centre is O.

*Cor.* 1. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; hence all great circles are equal.

*Cor.* 2. Two great circles always bisect each other; for their common intersection, passing through the centre, is a diameter.

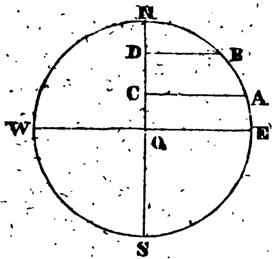
*Cor.* 3. Every great circle divides the sphere and its surface into two equal parts; for, if the two hemispheres were separated, and afterwards placed on the common base, with their convexities turned the same way, the two surfaces would exactly coincide, no point of the one being nearer the centre than any point of the other.

*Cor.* 4. The centre of a small circle, and that of the sphere, are in the same straight line perpendicular to the plane of the little circle.

*Cor.* 5. Small circles are the less, the farther they lie from the centre of the sphere; for the greater CO is, the less is the chord AB, the diameter of the small circle AMB.

*Cor.* 6. An arc of a great circle may always be made to pass through any two given points in the surface of the sphere; for the two given points and the centre of the sphere make three points, which determine the position of a plane. But if the two given points were at the extremities of a diameter, these two points and the centre would then lie in one straight line, and an infinite number of great circles might be made to pass through the two given points.

(101.) Let NESW represent a section of the earth, supposed to be a sphere, made by a plane passing through the axis of rotation, so that the circumference NESW shall correspond with a meridian. Let A and B be two different places on this meridian, whose latitudes are denoted by the arcs EA, EB. From A and B, draw the lines AC, BD perpendicular to the axis NS.

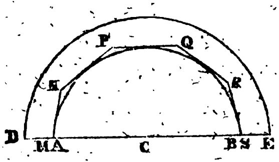


Now, if we conceive the semicircle NES to revolve about the diameter NS, the point E will describe the equator, while the points A and B will describe parallels of latitude; and as all circumferences are divided into 360°, it follows that the length of a degree of longitude at the latitude of A, is to a degree of longitude at the latitude B, as AC is to BD; *and, in general, degrees of longitude at different places of the earth, are as their distances from the axis of revolution.* The length of a degree of longitude at latitude 60°, is just one half the length of a degree of longitude at the equator.

### PROPOSITION IX.

**Theorem.** *The surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.*

It is first to be shown, that the diameter of a sphere, multiplied by the circumference of its great circle, can not measure the surface of a larger sphere. If possible, let AB × circ. AC be the surface of the sphere whose radius is CD.



About the circle whose radius is CA, circumscribe a regular polygon having an even number of sides, so as not to meet the circumference whose radius is CD : let M and S be the two opposite vertices of this polygon; and about the diameter MS, let the half polygon MPS be made to revolve. The surface described by this polygon will be measured (B. VIII, Prop. VII, Cor. 3) by MS × circ. AC; but MS is greater than AB; hence the surface described by this polygon is greater than AB × circ. AC, and consequently greater than the surface of the sphere whose radius is CD; but the surface of the sphere is greater than the surface described by the polygon, since the former envelopes the latter on all sides. Hence, in the first place, the diameter of a sphere multipled by the circumference of its great circle cannot measure the surface of a larger sphere.

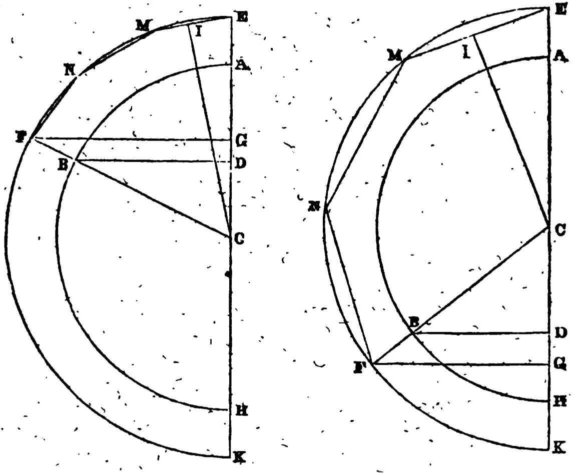
Neither can this same product measure the surface of a smaller sphere. For, if possible, let DE × circ. CD be the surface of that sphere whose radius is CA. The same construction being made as in the former case, the surface of the solid generated by the revolution of the half polygon will still be equal to MS × circ. AC; but MS is less than DE, and circ. AC is less than circ. CD; hence, for these two reasons, the surface of the solid described by the polygon must be less than DE × circ. CD, and therefore less than the surface of the sphere whose radius is AC; but the surface described by the polygon is greater than the surface of the sphere whose radius is AC, because the former envelopes the latter. Hence, in the second place, the diameter of a sphere multiplied by the circumference of its great circle cannot measure the surface of a smaller sphere.

Therefore the surface of a sphere is equal to its diameter multiplied by the circumference of its great circle.

*Cor.* The surface of the great circle is measured by multiplying its circumference by half the radius, or by a fourth of the diameter; hence *the surface of a sphere is four times that of its great circle.*

### PROPOSITION X.

THEOREM. *The surface of any spherical zone is equal to its altitude multiplied by the circumference of a great circle.*

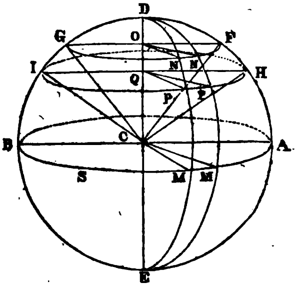


Let EF be any arc less or greater than a quadrant, and let FG be drawn perpendicular to the radius EC; the zone with one base, described by the revolution of the arc EF about EC, will be measured by EG × circ. EC.

For, suppose, first, that this zone is measured by something less; if possible, by EG × circ. CA. In the arc EF, inscribe a portion of a regular polygon EMNF, whose sides shall not reach the circumference described with the radius CA; and draw CI perpendicular to EM. The surface described by the polygon EMF turning about EC, will be measured by EG × circ. CI [B. VIII, Prop. vi, Cor. 2]. This quantity is greater than EG × circ. AC, which by hypothesis is the measure of the zone described by the arc EF. Hence the surface described by the polygon EMNF must be greater than the surface described by EF the circumscribed arc; whereas this latter surface is greater than the former, which it envelopes on all sides. Hence, in the first place, the measure of any spherical zone with one base cannot be less than the altitude multiplied by the circumference of a great circle.

Secondly, the measure of this zone cannot be greater than its altitude multiplied by the circumference of a great circle. For, suppose the zone described by the revolution of the arc AB about AC to be the proposed one; and, if possible, let zone AB > AD × circ. AC. The whole surface of the sphere composed of the two zones AB, BH is measured by AH × circ. AC [B. VIII, Prop. ix], or by AD × circ. AC + DH × circ. AC: hence, if we have zone AB > AD × circ. AC, we must also have zone BH > DH × circ. AC; which cannot be the case, as is shown above. Therefore, in the second place, the measure of a spherical zone with one base cannot be greater than the altitude of this zone multiplied by the circumference of a great circle.

Hence, finally, every spherical zone with one base is measured by its altitude multiplied by the circumference of a great circle.

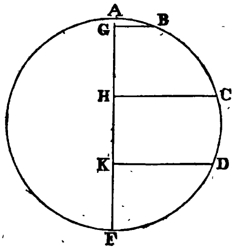


Let us now examine any zone with two bases, described by the revolution of the arc FH about the diameter DE. Draw FO, HQ perpendicular to this diameter. The zone described by the arc FH is the difference of the two zones described by the arcs DH and DF; the latter are respectively measured by DQ × circ. CD, and DO × circ. CD: hence the zone described by FH has for its measure

$$(DQ - DO) \times \text{circ. } CD, \text{ or } OQ \times \text{circ. } CD.$$

That is, any spherical zone, with one or two bases, is measured by its altitude multiplied by the circumference of a great circle.

*Cor.* Two zones, taken in the same sphere or in equal spheres, are to each other as their altitude; and any zone is to the surface of the sphere, as the altitude of that zone is to the diameter.



(102.) Let ABCDF be a section of the earth, made by a plane passing through its axis, which is represented by AF; also let B denote the place where the arctic circle cuts the meridian ABCDF; C and D, the corresponding points for the tropics; then will the line BG, perpendicular to the axis AF, be the radius of the arctic circle, and CH and DK, which are equal, will be the radii of the tropics. The line AG will denote the altitude of the frigid zone, GH the altitude of the temperate zone, and HK the altitude of the torrid zone : but surfaces of zones are to each other as their altitudes; therefore the frigid, temperate, and torrid zones are to each other as the lines AG, GH and HK.

If we assume the latitude of the arctic circle to be 66° 32′, and the tropics to be in latitude 23° 28′, we shall find the lines AG, GH and HK to be to each other as $1 - \cos 23° 28'$, $\cos 23° 28' - \sin 23° 28'$, $2 \sin 23° 28'$; or as 0·08271, 0·51907, 0·79644; or nearly as the numbers 4, 25, 38.

(103) **Problem.** *Suppose a person to be situated* h *miles above the surface of the earth, and let it be required to find what fractional part of the earth's entire surface is visible to him.*

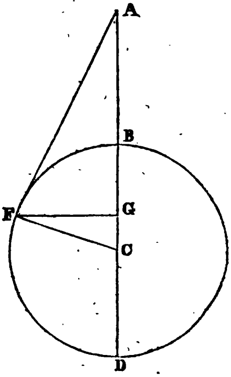
Let A denote the position of the observer. Draw AD passing through the centre of the earth, and AF tangent to its surface; also draw the radius FC, and FG perpendicular to AD.

Then the portion seen will be a zone having BG for its altitude, and this zone will be to the entire surface as BG is to BD (B. VIII, Prop. x, Cor.).

Calling R the radius of the earth, we have, by similar triangles ACF, CFG,

$$R+h : R :: R : GC;$$ therefore,

$$GC = \frac{R^2}{R+h}.$$ Consequently,



$$BG = R - GC = R - \frac{R^2}{R+h} = \frac{Rh}{R+h}.$$

Hence the fractional part of the earth's surface seen, is

$$\frac{Rh}{R+h} \div 2R = \frac{h}{2(R+h)}.$$

This fraction is always less than ½; which shows that it is impossible, from any point however distant, to see half the entire surface of a sphere.

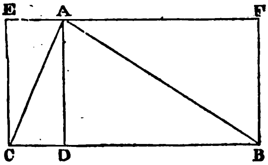
If $h = R$, the fraction becomes $\frac{1}{4}$; if $h = 2R$, it becomes $\frac{1}{3}$; if $h = \frac{2}{3}R$, it becomes $\frac{1}{5}$.

*Note.* In the above investigation, no allowance has been made for the refraction of light, which must of necessity modify the extent of surface visible.

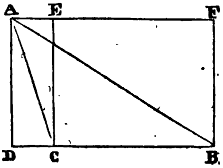
## PROPOSITION XI.

THEOREM. *If a triangle and a rectangle, having the same base and the same altitude, turn simultaneously about the common base, the solid described by the revolution of the triangle will be a third of the cylinder described by the revolution of the rectangle.*

Let ABC be the triangle, and CD the rectangle.



To the axis, draw the perpendicular AD; the cone described by the triangle ABD is the third part of the cylinder described by the rectangle AFBD (B. VIII, Prop. v, Cor.); also the cone described by the triangle ADC is the third part of the cylinder described by the rectangle ADCE: hence the sum of the two cones, or the solid described by ABC, is the third part of the two cylinders taken together, or of the cylinder described by the rectangle BCEF.



If the perpendicular AD falls without the triangle, the solid described by ABC will be the difference of the two cones described by ABD and ACD; but, at the same time, the cylinder described by BCEF will be the difference of the two cylinders described by AFBD and AECD. Hence the solid described by the revolution of the triangle will still be a third part of the cylinder described by the revolution of the rectangle having the same base and altitude.

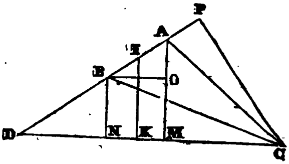
*Scholium.* The circle of which AD is radius has for its measure $\pi \times AD^2$ : hence $\pi \times AD^2 \times BC$ measures the cylinder described by BCEF, and $\frac{1}{3}\pi \times AD^2 \times BC$ measures the solid described the triangle ABC.

### PROPOSITION XII.

THEOREM. *If a triangle be revolved about a line drawn at pleasure through its vertex, the solid described by the triangle will have for its measure the area of the triangle multiplied by two-thirds of the circumference traced by the middle point of the base.*

Let CAB be the triangle, and CD the line about which it revolves.

Produce the side AB till it meets the axis CD in D; from the points A and B, draw AM, BN perpendicular to the axis.

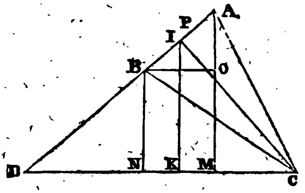


The solid described by the triangle CAD is measured (B. VIII, Prop. XI, Schol.) by $\frac{1}{3}\pi \times AM^2 \times CD$ ; the solid described by the triangle CBD is measured by $\frac{1}{3}\pi \times BN^2 \times CD$ : hence the difference of those solids, or the solid described by ABC, will have for its measure $\frac{1}{3}\pi\,(AM^2 - BN^2) \times CD$.

To this expression another form may be given. From I the middle point of AB, draw IK perpendicular to CD; and through B, draw BO parallel to CD: we shall have $AM + BN = 2\,IK$, and $AM - BN = AO$; hence $(AM+BN) \times (AM-BN)$, or $AM^2 - BN^2 = 2\,IK \times AO$. Hence the measure of the solid in question is expressed by $\frac{2}{3}\pi \times IK \times AO \times CD$. But if CP is drawn perpendicular to AB, the triangles ABO, DCP will be similar, and give the proportion

$$AO : CP :: AB : CD;$$

hence $AO \times CD = CP \times AB$, which $CP \times AB$ is double the area of the triangle ABC; hence we have ........ $AO \times CD = 2\,ABC$; hence the solid described by the triangle ABC is also measured by $\frac{4}{3}\pi \times ABC \times IK$, or, which is the same thing, by $ABC \times \frac{2}{3}$ circ. IK, circ. IK being equal to $2\pi \times IK$. Hence, *the solid described by the revolution of the triangle* ABC *has for its measure the area of this triangle multiplied by two-thirds of the circumference traced by* I *the middle point of the base.*

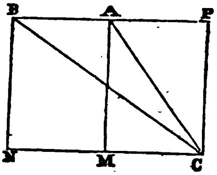


*Cor.* 1. If the side $AC = CB$, the line CI will be perpendicular to AB, the area ABC will be equal to $AB \times \frac{1}{2}\,CI$, and the solidity $\frac{4}{3}\pi \times ABC \times IK$ will become $\frac{2}{3}\pi \times AB \times IK \times CI$. But the triangles ABO, CIK are similar, and give the proportion

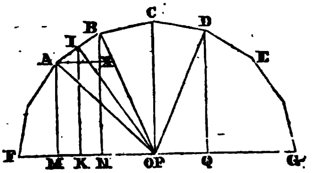
$$AB : BO \text{ or } MN :: CI : IK;$$

hence $AB \times IK = MN \times CI$: hence the solid described by the isosceles triangle ABC will have for its measure $\frac{2}{3}\pi \times MN \times CI^2$.

*Cor.* 2. The general solution appears to include the supposition that AB produced will meet the axis; but the results would be equally true, though AB were parallel to the axis.



Thus the cylinder described by AMNB is equal to $\pi . AM^2 . MN$; the cone described by ACM is equal to $\frac{1}{3} \pi . AM^2 . CM$, and the cone described by BCN to $\frac{1}{3} \pi . AM^2 . CN$. Add the first two solids, and take away the third: we shall have the solid described by ABC equal to $\pi . AM^2 . (MN + \frac{1}{3} CM - \frac{1}{3} CN)$; and since $CN - CM = MN$, this expression is reducible to ...... $\pi . AM^2 . \frac{2}{3} MN$, or $\frac{2}{3} \pi \, CP^2 . MN$, which agrees with the conclusion above drawn.



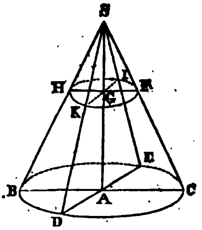
*Cor.* 3. Let AB, BC, CD be the several successive sides of a regular polygon, O its centre, and OI the radius of the inscribed circle; if the polygonal sector AOD lying all on one side of the diameter FG be supposed to perform a revolution about this diameter, the solid so described will have for its measure $\frac{2}{3} \pi . OI^2 . MQ$, MQ being that portion of the axis which is included by the extreme perpendiculars AM, DQ.

For, since the polygon is regular, all the triangles AOB, BOC, &c. are equal and isosceles. Now, by the last corollary, the solid produced by the isosceles triangle AOB has for its measure $\frac{2}{3} \pi . OI^2 . MN$; the solid described by the triangle BOC has for its measure $\frac{2}{3}\pi \,.\, OI^2 \,.\, NP$, and the solid described by the triangle COD has for its measure $\frac{2}{3}\pi \,.\, OI^2 \,.\, PQ$ : hence the sum of those solids, or the whole solid described by the polygonal sector AOD will have for its measure $\frac{2}{3}\pi \,.\, OI^2 \,.\, (MN + NP + IQ)$, or $\frac{2}{3}\pi \,.\, OI^2 \,.\, MQ$.

(104.) This proposition is a particular case of a more general theorem, called *Guldin's theorem*, or *the centrobarye method*, which is as follows :

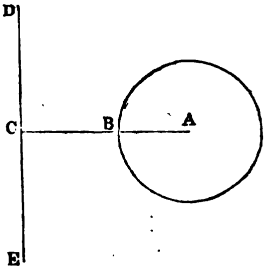
*The volume generated by the movement of a plane surface, so moved as that no two consecutive positions shall intersect, is measured by multiplying the area of the moving plane into the distance passed through by its centre of gravity.*

In the above proposition, we know that the centre of gravity of the triangle CAB is in the line CI, at $\frac{2}{3}$ its distance measured from C (Art. 73). Hence, the circumference described by this centre of gravity is $\frac{2}{3}$ of the circumference traced by the middle point I. Consequently the volume thus generated by the revolution of the triangle ABC has for its measure the area of this triangle multiplied by $\frac{2}{3}$ of the circumference traced by I, the middle point of the base.



(105.) Let us apply the method to find the solidity of the cone SBDC generated by the revolution of the right-angled triangle SAB about the side SA. The centre of gravity of the triangle SAB is $\frac{1}{3}$ of AB distant from the axis SA, and therefore moves through a distance equal to $\frac{2}{3}\pi AB$ : this, multiplied by the area of the triangle, which is $\frac{1}{2} AB \times AS$, gives $\frac{1}{3}\pi AB^2 \times AS$ for the volume of the cone, or, which is the same thing, $\pi AB^2 \times \frac{1}{3} AS$. That is, the solidity of a right cone may be found by multiplying its base by $\frac{1}{3}$ of its height.

(106.) Again, suppose the circle whose radius is AB to revolve about the line DE, thus generating a circular ring. Required the measure of this ring.



If we denote the distance AC by R, and the radius AB by $r$, we shall find $\pi r^2$ for the area of the circle, and $2\pi R$ for the distance passed through by its centre of gravity; hence the solid in question is $2\pi^2 r^2 R$.

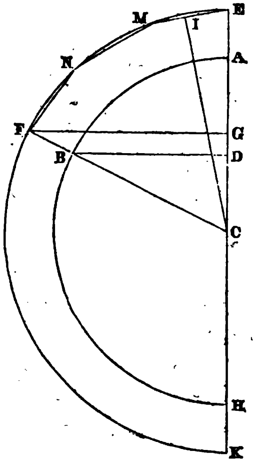
(107.) *The centrobaryc method* is also applicable to surfaces generated by the motion of a straight line, by multiplying the length of the line by the distance moved through by its centre of gravity.

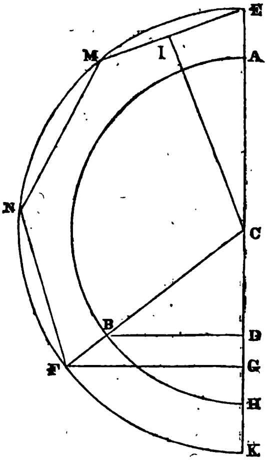
Thus, referring to the last figure, suppose we wish the surface of the ring generated by the revolution of the circle AB about the axis DE. While the circle generates the solid ring, its circumference generates the surface of the ring. The centre of gravity of the circumference is evidently at A, and moves through a distance denoted by $2\pi R$; this, multiplied by the circumference, which is denoted by $2\pi r$, gives $4\pi^2 R r$ for the surface of the ring.

While the right-angled triangle SAB (see figure on the preceding page), by its revolution about the side SA, generates the cone, the hypothenuse generates the cone's convex surface. The centre of gravity of SB is evidently at the middle point of SB; it therefore describes the circumference of a circle half as great as that of the base, and denoted by $\pi$ AB : this, multiplied by the length SB, gives $\pi$ AB × SB for the convex surface of the cone.

### PROPOSITION XIII.

THEOREM. *Every spherical sector is measured by the zone which forms its base, multiplied by a third of the radius; and the whole sphere has for its measure a third of the radius, multiplied by its surface.*





Let ABC be the circular sector, which, by its revolution about AC, describes the spherical sector; the zone described by AB being AD × circ. AC, or $2\pi . AC . AD$: and it is to be shown that this zone multiplied by $\frac{1}{3}$ of AC, or that $\frac{2}{3}\pi . AC^2 . AD$, will measure the sector.

First, suppose, if possible, that $\frac{2}{3}\pi . AC^2 . AD$ is the measure of a greater spherical sector, say of the spherical sector described by the circular sector ECF similar to ACB.

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In the arc EF, inscribe ECF a portion of a regular polygon, such that its sides shall not meet the arc AB; then imagine the polygonal sector ENFC to turn about EC, at the same time with the circular sector ECF. Let CI be a radius of the circle inscribed in the polygon, and let FG be drawn perpendicular to EC. The solid described by the polygonal sector will have for its measure $\frac{2}{3}\pi . CI^2 . EG$; but CI is greater than AC by construction, and EG is greater than AD; for, joining AB, EF, the similar triangles EFG, ABD give the proportion

$$EG : AD :: FG : BD :: CF : CB;$$

hence $EG > AD$. For this double reason, $\frac{2}{3}\pi . CI^2 . EG$ is greater than $\frac{2}{3}\pi . CA^2 . AD$. The first is the measure of the solid described by the polygonal sector; the second, by hypothesis, is that of the spherical sector described by the circular sector ECF: hence the solid described by the polygonal sector must be greater than the spherical sector; whereas, in reality, it is less, being contained in the latter. Hence our hypothesis was false: therefore, in the first place, the zone or base of a spherical sector multiplied by a third of the radius, cannot measure a greater spherical sector.

Secondly, it is to be shown that it cannot measure a less spherical sector. Let CEF be the circular sector, which, by its revolution, generates the given spherical sector; and suppose, if possible, that $\frac{2}{3}\pi . CE^2 . EG$ is the measure of some smaller spherical sector, say of that produced by the circular sector ACB.

The construction remaining as above, the solid described by the polygonal sector will still have for its measure

$\frac{2}{3}\pi . CI^2 . EG$. But CI is less than CE; hence the solid is less than $\frac{2}{3}\pi . CE^2 . EG$, which, according to the supposition, is the measure of the spherical sector described by the circular ACB : hence the solid described by the polygonal sector must be less than the spherical sector described by ACB; whereas, in reality, it is greater, the latter being contained in the former. Therefore, in the second place, it is impossible that the zone of a spherical sector, multiplied by a third of the radius, can be the measure of a smaller spherical sector.

Hence every spherical sector is measured by the zone which forms its base, multiplied by a third of the radius.

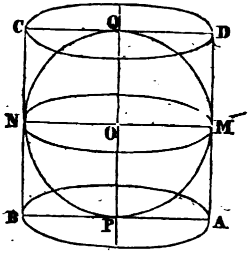
A circular sector ABC may increase till it becomes equal to a semicircle; in which case, the spherical sector described by its revolution is the whole sphere. Hence *the solidity of a sphere is equal to its surface multiplied by a third of the radius.*

*Cor.* The surfaces of spheres being as the squares of their radii, these surfaces multiplied by the squares of the radii must be as the cubes of the latter. Hence *the solidity of two spheres are as the cubes of their radii, or as the cubes of their diameters.*

*Schol.* Let R be the radius of a sphere : its surface will be $4\pi R^2$; its solidity, $4\pi R^2 \times \frac{1}{3}R$, or $\frac{4}{3}\pi . R^3$. If the diameter is named D, we shall have $R = \frac{1}{2}D$, and $R^3 = \frac{1}{8}D^3$: hence the solidity may likewise be expressed by $\frac{4}{3}\pi . \frac{1}{8}D^3$, or $\frac{1}{6}\pi D^3$.

### PROPOSITION XIV.

**Theorem.** *The surface of a sphere is to the whole surface of the circumscribed cylinder (including its bases), as 2 is to 3; and the solidities of these two bodies are to each other in the same ratio.*



Let MNPQ be a great circle of the sphere, and ABCD the circumscribed square. If the semicircle PMQ and the half square PADQ are at the same time made to revolve about the diameter PQ, the semicircle will generate the sphere, while the half-square will generate the cylinder circumscribed about that sphere.

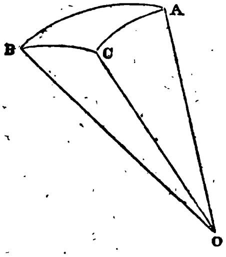
The altitude AD of that cylinder is equal to the diameter PQ; the base of the cylinder is equal to the great circle, its diameter AB being equal to MN: hence (B. VIII, Prop. IV), the convex surface of the cylinder is equal to the circumference of the great circle multiplied by its diameter. This measure (B. VIII, Prop. IX) is the same as that of the surface of the sphere: hence *the surface of the sphere is equal to the convex surface of the circumscribed cylinder.*

But the surface of the sphere is equal to four great circles; hence the convex surface of the cylinder is also equal to four great circles; and adding the two bases, each equal to a great circle, the total surface of the circumscribed cylinder will be equal to six great circles: hence the surface of the sphere is to the total surface of the circumscribed cylinder as 4 is to 6, or as 2 is to 3; which is the first branch of the proposition.

In the next place, since the base of the circumscribed cylinder is equal to a great circle, and its altitude to the diameter, the solidity of the cylinder (B. VIII, Prop. I) will be equal to a great circle multiplied by its diameter. But (B. VIII, Prop. XIII) the solidity of the sphere is equal to four great circles multiplied by a third of the radius; in other terms, to one great circle multiplied by $\frac{4}{3}$ of the radius, or by $\frac{2}{3}$ of the diameter: hence the sphere is to the circumscribed cylinder as 2 to 3, and consequently the solidities of these two bodies are as their surfaces.

### PROPOSITION XV.

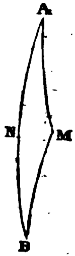
THEOREM. *In every spherical triangle, any side is less than the sum of the other two.*



Let O be the centre of the sphere; and draw the radii OA, OB, OC. Imagine the planes AOB, AOC, COB; those planes will form a solid angle at the point O; and the angles AOB, AOC, COB will be measured by AB, AC, BC the sides of the spherical triangle. But (B. VI, Prop. XIX) each of the three plane triangles composing a solid angle is less than the sum of the other two: hence any side of the triangle ABC is less than the sum of the other two.

### PROPOSITION XVI.

THEOREM. *The shortest distance between one point and another, on the surface of a sphere, is the arc of the great circle which joins the two given points.*

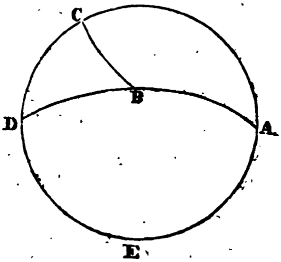


Let ANB be the arc of the great circle which joins the points A and B; and without this line, if possible, let M be a point in the line of the shortest distance between A and B. Through the point M draw MA, MB, arcs of great circles; and take BN = MB.

By the last proposition, the arc ANB is shorter than AM+MB: take BN respectively from both; there will remain AN < AM. Now, the distance of B from M, whether it be the same with the arc BM or with any other line, is equal to the distance of B from N; for, by making the plane of the great circle BM to revolve about the diameter which passes through B, the point M may be brought into the position of the point N; and the shortest line between M and B, whatever it may be, will then be identical with that between N and B: hence the two lines from A to B, one passing through M, the other through N, have an equal part in each, the part from M to B equal to the part from N to B. The first line is the shorter, by hypothesis; hence the distance from A to M must be shorter than the distance from A to N; which is absurd, the arc AM being proved greater than AN. Hence no point of the shortest line from A to B can lie out of the arc ANB; consequently this arc is itself the shortest distance between its two extremities.

### PROPOSITION XVII.

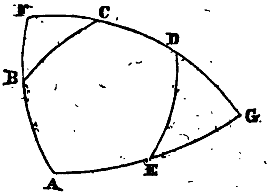
**Theorem.** *The sum of all the three sides of a spherical triangle is less than the circumference of a great circle.*



Let ABC be any spherical triangle: produce the sides AB, AC till they meet again in D. The arcs ABD, ACD will be semicircumferences, since (B. 8, Prop. 8, Cor. 2) two great circles always bisect each other. But in the triangle BCD we have (B. VIII, Prop. xv) the side BC<BD+CD : add AB+AC to each; we shall have AB+AC+BC < ABD+ACD, that is to say, less than a circumference.

### PROPOSITION XVIII.

**Theorem.** *The sum of all the sides of any spherical polygon is less than the circumference of a great circle.*

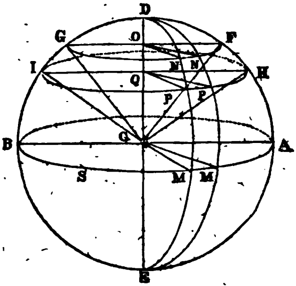


Let us take, for example, the pentagon ABCDE. Produce the sides AB, DC till they meet in F; then since BC is less than BF+CF, the perimeter of the pentagon ABCDE will be less than that of the quadrilateral AEDF. Again, produce the sides AE, FD till they meet in G; we shall have ED<EG+DG : hence the perimeter of the quadrilateral AEDF is less

than that of the triangle AFG, which last is itself less than the circumference of a great circle : hence the perimeter of the polygon ABCDE is less than this same circumference.

### PROPOSITION XIX.

**Theorem.** *If a diameter be drawn perpendicular to the plane of a great circle, its extremities will be the poles of that circle, and also of all small circles parallel to it.*



For, DC being perpendicular to the plane AMB, is perpendicular to all the straight lines CA, CM, CB, &c. drawn through its foot in this plane ; hence all the arcs DA, DM, DB, &c. are quarters of the circumference. So likewise are all the arcs EA, EM, EB, &c. : hence the points D and E are each equally distant from all the points of the circumference AMB ; therefore (Def. 11) they are the poles of that circumference.

Again, the radius DC, perpendicular to the plane AMB, is perpendicular to its parallel FNG ; hence (B. VIII, Prop. viii, Cor. 4) it passes through O the centre of the circle FNG : therefore, if the oblique lines DF, DN, DG be drawn, these oblique lines will diverge equally from the perpendicular DO, and will themselves be equal ; but, the chords being equal, the arcs are equal : hence the point D is the pole of the small circle FNG ; and, for like reasons, the point E is the other pole.

*Cor.* 1. Every arc DM drawn from a point in the arc of a great circle AMB to its pole, is a quarter of the circumference, which, for the sake of brevity, is usually named *a quadrant;* and this quadrant, at the same time, makes a right angle with the arc AM. For (B. VI, Prop. XVI) the line DC being perpendicular to the plane AMC, every plane DMC passing through the line DC is perpendicular to the plane AMC; hence the angle of these planes, or the angle AMD, is a right angle.

*Cor.* 2. To find the pole of a given arc AM, draw the indefinite arc MD perpendicular to AM; take MD equal to a quadrant: the point D will be one of the poles of the arc AMD. Or thus, at the two points A and M, draw the arcs AD and MD perpendicular to AM; their point of intersection D will be the pole required.

*Cor.* 3. Conversely, if the distance of the point D from each of the points A and M be equal to a quadrant, the point D will be the pole of the arc AM; and also the angles DAM, AMD will be right angles.

For, let C be the centre of the sphere; and draw the radii CA, CD, CM. Since the angles ACD, MCD are right, the line CD is perpendicular to the two straight lines CA, CM; it is therefore perpendicular to their plane: hence the point D is the pole of the arc AM, and consequently the angles DAM, AMD are right.

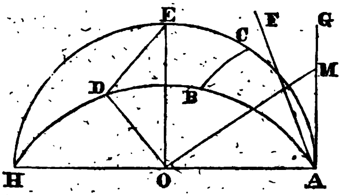
*Scholium.* The properties of these poles enable us to describe arcs of a circle on the surface of a sphere, with the same facility as on a plane surface. It is evident, for instance, that by turning the arc DF, or any other line extending to the same distance, round the point D, the extremity F will describe the small circle FNG; and by turning the quadrant DFA round the point D, its extremity A will describe the arc of the great circle AM.

If the arc AM were required to be produced, and nothing were given but the points A and M through which it was to pass, we should first have to determine the pole D, by the intersection of two arcs described from the points A and M as centres, with a distance equal to a quadrant; the pole D being found, we might describe the arc AM and its prolongation, from D as a centre, and with the same distance as before.

Lastly, if it be required from a given point P to draw a perpendicular on the given arc AM; produce this arc to S, till the distance PS be equal to a quadrant; then from the pole S, and with the same distance, describe the arc PM, which will be the perpendicular required.

## PROPOSITION XX.

THEOREM. *Every plane perpendicular to a radius at its extremity, is a tangent to the sphere.*

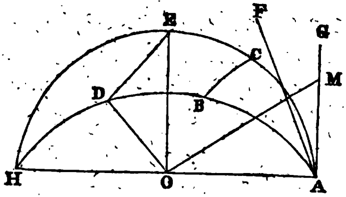


Let FAG be a plane perpendicular to the radius OA. Any point M in this plane being assumed, and OM, AM being joined, the angle OAM will be right, and hence the distance OM will be greater than OA. Hence the point M lies without the sphere; and as the case is similar with every other point in the plane FAG, this plane can have no point but A common to it with the surface of the sphere: it is therefore a tangent (Def. 10).

*Scholium.* In the same way it may be shown that two spheres have but one point in common, and therefore touch each other, when the distance between their centres is equal to the sum or the difference of their radii; in which case, the centres and the point of contact lie in the same straight line.

### PROPOSITION XXI.

THEOREM. *The angle formed by two arcs of great circles is equal to the angle formed by the tangents of these arcs at the point of intersection, and is therefore measured by the arc described from the point of intersection as a pole, between the sides, produced if necessary.*

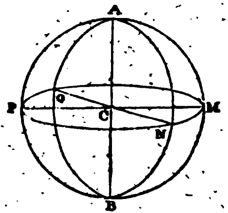


For the tangent AF drawn in the plane of the arc AB, is perpendicular to the radius AO; and the tangent AG drawn in the plane of the arc AC, is perpendicular to the same radius AO: hence (B. VI, Def. 4) the angle FAG is equal to the angle contained by the planes OAB, OAC, which is that of the arcs AB, AC, and is named BAC.

In like manner, if the arcs AD and AE are both quadrants, the lines OD, OE will be perpendicular to AO, and the angle DOE will still be equal to the angle of the planes AOD, AOE; hence the arc DE is the measure of the angle contained by these planes, or of the angle CAB.

*Cor.* The angles of spherical triangles may be compared together, by means of the arcs of great circles described from their vertices as poles, and included between their sides : hence it is easy to make an angle of this kind equal to a given angle.

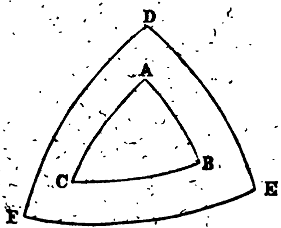
*Schol.* Vertical angles, such as ACO and BCN are equal; for either of them is still the angle formed by the two planes ACB, OCN.



It is farther evident, that, in the intersection of two arcs ACB, OCN, the two adjacent angles ACO, OCB taken together are equal to two right angles.

### PROPOSITION XXII.

**Theorem.** *If, with the vertices of a given triangle as poles, arcs be described, forming a new triangle, then will the vertices of this new triangle be the poles respectively of the sides of the given triangle.*

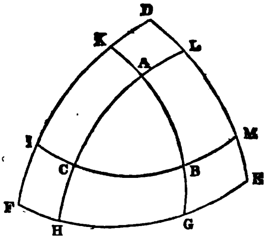


For, the point A being the pole of the arc EF, the distance AE is a quadrant; the point C being the pole of the arc DE, the distance CE is likewise a quadrant : hence the point E is removed the length of a quadrant from each of the points A and C; hence (B. VIII, Prop. XIX, Cor. 3) it is the pole of the arc AC. It might be shown by the same method, that D is the pole of the arc BC, and F that of the arc AB.

*Cor.* Hence the triangle ABC may be described by means of DEF, as DEF may by means of ABC.

## PROPOSITION XXIII.

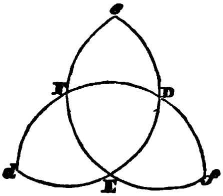
THEOREM. *The same supposition being made as in the last proposition, each angle in either one of the triangles will be measured by a semicircumference* MINUS *the side lying opposite to it in the other triangle.*



Produce the sides AB, AC, if necessary, till they meet EF in G and H. The point A being the pole of the arc GH, the angle A will be measured by that arc; but the arc EH is a quadrant, and likewise GF, E being the pole of AH, and F of AG: hence $EH+GF$ is equal to a semicircumference. Now, $EH+GF$ is the same as $EF+GH$; hence the arc GH, which measures the angle A, is equal to a semicircumference *minus* the side EF. In like manner, the angle B will be measured by $\frac{1}{2}$ circ.$-DF$; the angle C, by $\frac{1}{2}$ circ.$-DE$.

And this property must be reciprocal in the two triangles, since each of them is described in a similar manner by means of the other. Thus we shall find the angles D, E, F of the triangle DEF to be measured respectively by $\frac{1}{2}$ circ.$-BC$, $\frac{1}{2}$ circ.$-AC$, $\frac{1}{2}$ circ.$-AB$. The angle D, for example, is measured by the arc MI; but

$MI + BC = MC + BI = \frac{1}{2}$ circ.: hence the arc MI, the measure of D, is equal to $\frac{1}{2}$ circ.$-BC$; and so of all the rest.

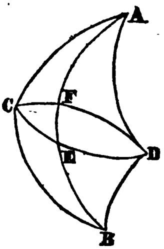


*Scholium.* It must farther be observed, that besides the triangle DEF, three others might be formed by the intersection of the three arcs DE, EF, DF. But the proposition immediately before us is applicable only to the central triangle, which is distinguished from the other three by the circumstance that the two angles A and D lie on the same side of BC, the two B and E on the same side of AC, and the two C and F on the same side of AB.

Various names have been given to the triangles ABC, DEF; but they are now more generally denominated *polar triangles.*

### PROPOSITION XXIV.

THEOREM. *Any triangle on a sphere being given, another triangle may be constructed, which shall have all its parts equal respectively to the corresponding parts of the given triangle.*



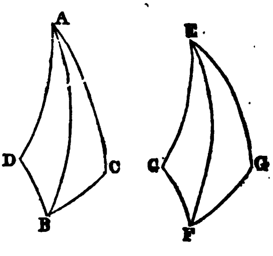
Let ABC be the given triangle. With A as a pole, describe the arc DEC passing through C; and with B as a pole, describe another arc DFC passing through C, and intersecting the former arc at C. Then will the triangle ADB have all its parts equal to the corresponding parts of the triangle ABC.

For, by construction, the side AD=AC, DB=BC, and AB is common; hence those two triangles have their sides equal each to each, and it is to be shown that the angles opposite these equal sides are also equal.

If the centre of the sphere is supposed to be at O, a solid angle may be conceived as formed at O by the three plane angles AOB, AOC, BOC; likewise another solid angle may be conceived as formed by the three plane angles AOB, AOD, BOD. And because the sides of the triangle ABC are equal to those of the triangle ADB, the plane angles forming the one of these solid angles must be equal to the plane angles forming the other, each to each; but in this case the planes, in which the equal angles lie, are equally inclined to each other; hence all the angles of the spherical triangle DAB are respectively equal to those of the triangle CAB, namely, DAB=BAC, DBA=ABC, and ADB=ACB: therefore the sides and the angles of the triangle ADB are equal to the sides and the angles of the triangle ACB.

### PROPOSITION XXV.

THEOREM. *Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when they have each an equal angle included between equal sides.*



Suppose the side AB=EF, the side AC=EG, and the angle BAC=FEG; the triangle EFG may be placed on the triangle ABC, or on ABD symmetrical with ABC, just as two rectilineal triangles are placed upon each other when they have an equal angle included between equal sides. Hence all the parts of the triangle EFG will be equal to all the parts of the triangle ABC; that is, besides the three parts equal by hypothesis, we shall have the side BC = FG, the angle ABC = EFG, and the angle ACB = EFG.

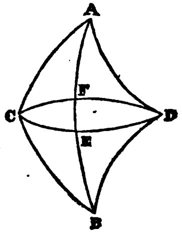
### PROPOSITION XXVI.

THEOREM. *Two triangles on the same sphere, or on equal spheres, are equal in all their parts, when two angles and the included side of the one are equal to two angles and the included side of the other.*

For one of those triangles, or the triangle symmetrical with it, may be placed on the other, and be made to coincide with it, as is obvious.

### PROPOSITION XXVII.

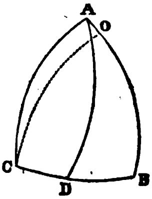
THEOREM. *If two triangles on the same sphere, or on equal spheres, have all their sides respectively equal, their angles will likewise be all respectively equal, the equal angles lying opposite the equal sides.*



The truth is evident by Prop. XXIV, Book VIII, where it was shown that, with three given sides AB, AC, BC, there can only be two triangles ACB, ABD, different as to the position of their parts, and equal as to the magnitude of those parts. Hence those two triangles, having all their sides respectively equal in both, must either be absolutely equal, or at least *symmetrically* so; in both of which cases, their corresponding angles must be equal, and lie opposite to equal sides.

### PROPOSITION XXVIII.

THEOREM. *In every isosceles spherical triangle, the angles opposite the equal sides are equal; and, conversely, if two angles of a spherical triangle are equal, the triangle will be isosceles.*



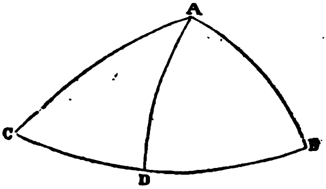
*First.* Suppose the side AB=AC; we shall have the angle C=B. For, if the arc AD be drawn from the vertex A to the middle point D of the base, the two triangles ABD, ACD will have all the sides of the one respectively equal to the corresponding sides of the other, namely, AD common, BD=DC, and AB=AC: hence, by the last proposition, their angles will be equal; therefore B=C.

*Secondly.* Suppose the angle B=C; we shall have the side AC=AB. For, if not, let AB be the greater of the two; take BO=AC, and join OC. The two sides BO, BC are equal to the two AC, BC; the angle OBC contained by the first two, is equal to ACB contained by the second two. Hence (B. VIII, Prop. xxv) the two triangles BOC, ACB have all their other parts equal; hence the angle OCB=ABC; but, by hypothesis, the angle ABC=ACB; hence we have OCB=ACB, which is absurd; therefore AB is not different from AC; that is, the sides AB, AC, opposite to the equal angles B and C, are equal.

*Scholium.* The same demonstration proves that the angle BAD=DAC, and the angle BDA=ADC. Hence the two last are right angles; consequently *the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base, is at right angles to that base, and bisects the opposite angle.*

### PROPOSITION XXIX.

THEOREM. *In any spherical triangle, the greater side is opposite the greater angle; and, conversely, the greater angle is opposite the greater side.*



*First*. Suppose the angle $A > B$ : make the angle $BAD = B$; then (B. VIII, Prop. XXVIII) we shall have $AD=DB$; but $AD+DC$ is greater than AC : hence, putting DB in place of AD, we shall have $DB+DC$ or $BC > AC$.

*Secondly*. If we suppose $BC>AC$, the angle BAC will be greater than ABC. For, if BAC were equal to ABC, we should have $BC=AC$; if BAC were less than ABC, we should then, as has just been shown, find $BC < AC$. Both these conclusions are false; hence the angle BAC is greater than ABC.

### PROPOSITION XXX.

THEOREM. *If two triangles on the same sphere, or on equal spheres, are mutually equiangular, they will also be mutually equilateral.*

Let A and B be the two given triangles; P and Q their polar triangles. Since the angles are equal in the triangles A and B, the sides will be equal in the polar triangles P and Q (B. VIII, Prop. XXIII); but since the triangles P and Q are mutually equilateral, they must also (B. VIII, Prop. XXVII) be mutually equiangular; and, lastly, the angles being equal in the triangles P and Q, it follows

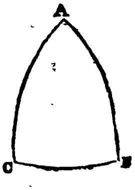
(B. VIII, Prop. XXIII) that the sides are equal in their polar triangles A and B. Hence the mutually equiangular triangles A and B are at the same time mutually equilateral.

*Schol.* This proposition is not applicable to rectilineal triangles; in which, equality among the angles indicates only proportionality among the sides. Nor is it difficult to account for the difference observable, in this respect, between spherical and rectilineal triangles. In the proposition now before us, as well as in the three last, which treat of the comparison of triangles, it is expressly required that the arcs be traced on the same sphere, or on equal spheres. Now similar arcs are to each other as their radii: hence, on equal spheres, two triangles cannot be similar without being equal; therefore it is not strange that equality among the angles should produce equality among the sides.

The case would be different, if the triangles were drawn upon unequal spheres; there, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of their spheres.

### PROPOSITION XXXI.

**THEOREM.** *The sum of all the angles in any spherical triangle is less than six right angles, and greater than two.*



For, in the first place, every angle of a spherical triangle is less than two right angles (see the following scholium): hence the sum of all the three is less than six right angles.

Secondly, the measure of each angle

in a spherical triangle (B. VIII, Prop. XXIII) is equal to the semicircumference *minus* the corresponding side of the polar triangle ; hence the sum of all the three is measured by three semicircumferences *minus* the sum of all the sides of the polar triangle. Now (B. VIII, Prop. XVII) this latter sum is less than a circumference ; therefore, taking it away from three semicircumferences, the remainder will be greater than one semicircumference, which is the measure of two right angles. Hence, in the second place, the sum of all the angles in a spherical triangle is greater than two right angles.

*Cor.* 1. The sum of all the angles in a spherical triangle is not constant, like that of all the angles in a rectilineal triangle ; it varies between two right angles and six, without ever arriving at either of these limits. Two given angles, therefore, do not serve to determine the third.

*Cor.* 2. A spherical triangle may have two or even three angles right, two or three obtuse.

If the triangle ABC have two right angles B and C, the vertex A will (B. VIII, Prop. XIX) be the pole of the base BC ; and the sides AB, AC will be quadrants.

If the angle A is also right, the triangle ABC will have all its angles right, and its sides quadrants. The tri-rectangular triangle is contained eight times in the surface of the sphere ; as is evident from the figure in the next proposition, supposing the arc MN to be a quadrant.

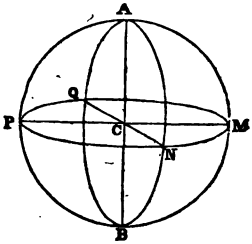
*Scholium.* In all the preceding observations, we have supposed, in conformity with Def. 12, that our spherical triangles have always each of their sides less than a semicircumference; from which it follows that any one of their angles is always less than two right angles. For (see the figure of Prop. XVII) if the side AB is less than a semicircumference, and AC is so likewise, both those arcs will require to be produced before they can meet in D. Now the two angles ABC, CBD, taken together, are equal to two right angles: hence the angle ABC itself is less than two right angles.

We may observe, however, that some spherical triangles do exist, in which certain of the sides are greater than a semicircumference, and certain of the angles greater than two right angles. Thus, if the side AC is produced so as to form a whole circumference ACDE, the part which remains after subtracting the triangle ABC from the hemisphere, is a new triangle also designated by ABC, and having AB, BC, AEDC for its sides. Here, it is plain, the side AEDC is greater than the semicircumference AED; and, at the same time, the angle B opposite to it exceeds two right angles by the quantity CBD.

The triangles whose sides and angles are so large have been excluded from our definition; but the only reason was, that the solution of them, or the determination of their parts, is always reducible to the solution of such triangles as are comprehended by the definition. Indeed, it is evident enough, that if the sides and angles of the triangle ABC are known, it will be easy to discover the angles and sides of the triangle which bears the same name, and is the difference between a hemisphere and the former triangle.

## PROPOSITION XXXII.

**Theorem.** ***The surface of a lune is to the entire surface of the sphere, as the angle of the lune is to four right angles, or as the arc which measures the angle of the lune is to the circumference.***



Suppose, in the first place, the arc MN to be to the circumference MNPQ as some one rational number is to another, as 5 to 48, for example. The circumference MNPQ being divided into 48 equal parts, MN will contain 5 of them; and if the pole A were joined with the several points of division, by as many quadrants, we should in the hemisphere AMNPQ have 48 triangles, all equal, because having all their parts equal. Hence the whole sphere must contain 96 of those partial triangles, and the lune AMBNA will contain 10 of them: hence the lune is to the sphere as 10 is to 96, or as 5 to 48; in other words, as the arc MN is to the circumference.

If the arc MN is not commensurable with the circumference, we may still show, by the mode of reasoning employed in the case of incommensurable magnitudes, that in this instance, also, the lune is to the sphete as MN is to the circumference.

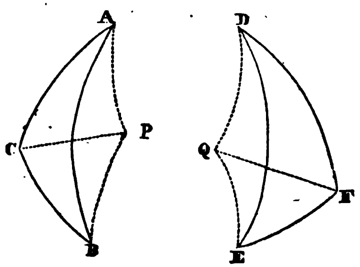
*Cor.* 1. Two lunes are to each other as their respective angles.

*Cor.* 2. It was shown (B. VIII, Prop. XXXI, Cor. 2) that the whole surface of the sphere is equal to eight tri-rectangular triangles : hence, if the area of one such triangle is taken for unity, the surface of the sphere will be represented by 8. This granted, the surface of the lune whose angle is A will be expressed by 2 A (the angle A being always estimated from the right angle assumed as unity), since 2 A : A : : 8 : 4. Thus we have here two different unities : one for angles, being the right angle; the other for surfaces, being the tri-rectangular spherical triangle, or the triangle whose angles are all right, and whose sides are quadrants.

*Scholium.* The spherical ungula bounded by the planes AMB, ANB, is to the whole solid sphere, as the angle A is to four right angles; for, the lunes being equal, the spherical ungulas will also be equal : hence two spherical ungulas are to each other as the angles formed by the planes which bound them.

### PROPOSITION XXXIII.

**Theorem.** *Two symmetrical spherical triangles are equal in surface.*



Let ABC, DEF be two symmetrical triangles, that is to say, two triangles having their sides

AB = DE,
AC = DF,
CB = EF,

and yet incapable

of coinciding with each other : we are to show that the surface ABC is equal to the surface DEF.

Let P be the pole of the little circle passing through the three points A, B, C ; from this point, draw (B. VIII, Prop. XIX, Schol.) the equal arcs PA, PB, PC ; at the point F, make the angle DFQ=ACP, the arc FQ=CP ; and join DQ, EQ.

The sides DF, FQ are equal to the sides AC, CP ; the angle DFQ=ACP : hence (B. VIII, Prop. XXV) the two triangles DFQ, ACP are equal in all their parts ; hence the side DQ=AP, and the angle DQF=APC.

In the proposed triangles DFE, ABC, the angles DFE, ACB opposite to the equal sides DE, AB being equal (B. VIII, Prop. XXIV), if the angles DFQ, ACP which are equal by construction, be taken away from them, there will remain the angle QFE equal to PCB. Also the sides QF, FE are equal to the sides PC, CB ; hence the two triangles FQE, CPB are equal in all their parts : hence the side QE=PB, and the angle FQE=CPB.

Now, observing that the triangles DFQ, ACP, which have their sides respectively equal, are at the same time isosceles, we shall see them to be capable of mutual adaptation when applied to each other ; for, having placed PA on its equal QF, the side PC will fall on its equal QD, and thus the two triangles will exactly coincide ; hence they are equal, and the surface DQF=APC. For a like reason, the surface FQE = CPB, and the surface DQE=APB : hence we have

$$\text{DFQ}+\text{FQE}-\text{DQE} = \text{APC}+\text{CPB}-\text{APB},$$

or

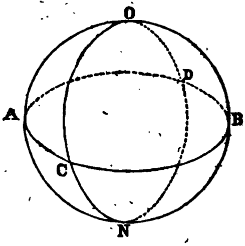
$$\text{DFE} = \text{ABC};$$

therefore the two symmetrical triangles ABC, DEF are equal in surface.

*Schol.* The poles P and Q might lie within the triangles ABC, DEF; in which case, it would be requisite to add the three triangles DQF, FQE, DQE together, in order to make up the triangle DEF; and, in like manner, to add the three triangles APC, CPB, APB together, in order to make up the triangle ABC. In all other respects, the demonstration and the result would still be the same.

### PROPOSITION XXXIV.

THEOREM. *If two great circles intersect each other on the surface of a hemisphere, the sum of the opposite triangles thus formed will be equivalent to the lune whose angle is equal to the angle formed by the circles.*

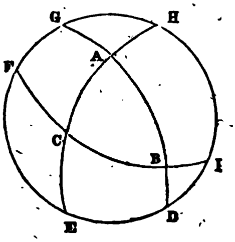


Let the circumferences AOB, COD intersect on the hemisphere OACBD; then will the opposite triangles AOC, BOD be equal to the lune whose angle is BOD.

For, producing the arcs OB, OD in the other hemisphere, till they meet in N; the arc OBN will be a semicircumference, and AOB one also; and taking OB from each, we shall have BN = AO. For a like reason, we have DN=CO, and BD=AC. Hence the two triangles AOC, BDN have their three sides respectively equal; besides, they are so placed as to be symmetrical: hence (B. VIII, Prop. XXXIII) they are equal in surface, and the sum of the triangles AOC, BOD is equal to the lune OBNDO whose angle is BOD.

### PROPOSITION XXXV.

THEOREM. *The surface of any spherical triangle is measured by the excess of the sum of its three angles above two right angles.*



Let ABC be the proposed triangle : produce its sides till they meet the great circle DEFG, drawn anywhere without the triangle. By the last proposition, the two triangles ADE, AGH are together equal to the lune whose angle is A, and which is measured (B. VIII, Prop. XXXII, Cor. 2) by 2 A : hence we have ADE+AGH = 2 A ; and, for a like reason, BGF+BID = 2 B, and CIH+CFE = 2 C. But the sum of those six triangles exceeds the hemisphere by twice the triangle ABC, and the hemisphere is represented by 4 ; therefore twice the triangle ABC is equal to 2 A + 2 B + 2 C — 4, and consequently once ABC = A + B + C — 2 : hence, every spherical triangle is measured by the sum of all its angles *minus* two right angles.

*Cor.* 1. However many right angles there be contained in this measure, just so many tri-rectangular triangles, or eighths of the sphere, which (B. VIII, Prop. XXXII, Cor. 2) are the unit of surface, will the proposed triangle contain. If the angles, for example, are each equal to $\frac{4}{3}$ of a right angle, the three angles will amount to 4 right angles, and the proposed triangle will be represented by 4 — 2 or 2 ; therefore it will be equal to two tri-rectangular triangles, or to the fourth part of the whole surface of the sphere.

*Cor.* 2. The spherical triangle ABC is equal to the lune whose angle is $\frac{A+B+C}{2} - 1$. Likewise the spherical pyramid which has ABC for its base, is equal to the spherical ungula whose angle is $\frac{A+B+C}{2} - 1$.

*Schol.* While the spherical triangle ABC is compared with the tri-rectangular triangle, the spherical pyramid, which has ABC for its base, is compared with the tri-rectangular pyramid, and the same ratio is found to subsist between them. The solid angle at the vertex of the pyramid is, in like manner, compared with the solid angle at the vertex of the tri-rectangular pyramid. These comparisons are founded on the coincidence of the corresponding parts. If the bases of the pyramids coincide, the pyramids themselves will evidently coincide, and likewise the solid angles at their vertices. From this, the following consequences are deduced :

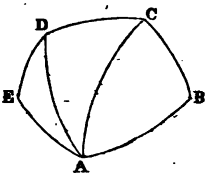
*First.* Two triangular spherical pyramids are to each other as their bases ; and since a polygonal pyramid may always be divided into a certain number of triangular ones, it follows that any two spherical pyramids are to each other, as the polygons which form their bases.

*Second.* The solid angles at the vertices of those pyramids are also as their bases : hence, for comparing any two solid angles, we have merely to place their vertices at the centres of two equal spheres, and the solid angles will be to each other as the spherical polygons intercepted between their planes or faces (B. VI, Prop. XXI, Schol.).

The vertical angle of the tri-rectangular pyramid is formed by three planes at right angles to each other : this angle, which may be called *a right solid angle*, will serve as a very natural unit of measure for all other solid angles; and, if so, the same number that exhibits the area of a spherical polygon, will exhibit the measure of the corresponding solid angle. If the area of the polygon is $\frac{3}{4}$, for example, in other words, if the polygon is $\frac{3}{4}$ of the trirectangular polygon, then the corresponding solid angle will also be $\frac{3}{4}$ of the right solid angle.

### PROPOSITION XXXVI.

THEOREM. *The surface of a spherical polygon is measured by the sum of all its angles,* minus *the product of two right angles by the number of sides in the polygon* minus *two.*



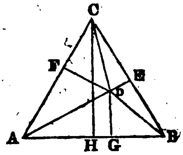
From one of the vertices A, let diagonals AC, AD be drawn to all the other vertices; the polygon ABCDE will be divided into as many triangles, *minus* two, as it has sides. But the surface of each triangle is measured by the sum of all its angles *minus* two right angles, and the sum of the angles in all the triangles is evidently the same as that of all the angles in the polygon: hence the surface of the polygon is equal to the sum of all its angles, diminished by twice as many right angles as it has sides *minus* two.

*Schol.* Let $s$ be the sum of all the angles in a spherical polygon, and $n$ the number of its sides; the right angle being taken for unity, the surface of the polygon will be measured by $s-2(n-2)$, or $s-2n+4$.

# APPENDIX.

## APPLICATION OF ALGEBRA TO THE SOLUTION OF GEOMETRICAL PROBLEMS.

Problem I. *In an equilateral triangle, having given the lengths of the three perpendiculars drawn from a certain point within it to the three sides, to determine its side.*



Let ABC be the equilateral triangle, and DE, DF, DG the perpendiculars from the point D upon the sides respectively. Denote these perpendiculars by $a$, $b$, $c$, in order, and the side of the triangle ABC by $2x$. Then, if the perpendicular CH be drawn, $\text{CH} = \sqrt{\text{AC}^2 - \text{AH}^2} = \sqrt{4x^2 - x^2} = x\sqrt{3}$.

The area of the triangle $\text{ADB} = \frac{1}{2}\text{AB}.\text{GD} = cx$. Similarly the triangle $\text{BDC} = ax$, the triangle $\text{GDA} = bx$, and the triangle $\text{ACB} = \frac{1}{2}\text{AB}.\text{CH} = x^2\sqrt{3}$. Also $\text{BDC} + \text{CDA} + \text{ADB} = \text{ABC}$; that is, in symbols, $x^2\sqrt{3} = (a+b+c)x$, and $x = \dfrac{a+b+c}{\sqrt{3}}$, which is half the side of the triangle sought.

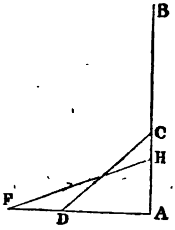
*Cor.* From the resulting equation, we have

$$x\sqrt{3} = a + b + c:$$

we also had $\text{CH} = x\sqrt{3}$. Hence $\text{CH} = a + b + c$; or the whole perpendicular CH is equal to the sum of the three smaller perpendiculars from D upon the sides, whenever the point D is taken *within* the triangle. Had the point D been taken *without* the triangle, the perpendicular upon the side which subtends the angle within which the point lies would become negative. Thus, had the point been without the triangle, but between the sides AB, AC *produced*, then CH = DF + DG − DE.

**Problem II.** *A maypole was broken off by the wind, and its top struck the ground twenty feet from the base; and, being repaired, was broken a second time five feet lower, and its top struck the ground ten feet farther from the base. What was the height of the maypole?*

Let AB be the unbroken maypole, C and H the points in which it was successively broken, and D and F the corresponding points at which the top B struck the ground. Then will CAD and HAF be right-angled triangles.



Put $BC = CD = x$, $CA = y$, $AD = a$, $AF = b$, and $CH = c$. Then $AB = x+y$, $BH = HF = x+c$, and $HA = y-c$; therefore [B. II, Prop. VIII] we have

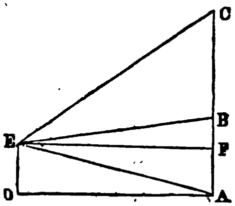
$$y^2 + a^2 = x^2, \qquad [1]$$

$$(y-c)^2 + b^2 = (x+c)^2. \qquad [2]$$

Expanding [2] and subtracting [1] from it, we have, after a slight reduction,

$$x+y = \frac{b^2-a^2}{2c} = 50 \text{ feet, the required height.}$$

**Problem** III. *A statue eighty feet high stands on a pedestal fifty feet high, and, to a spectator on the horizontal plane, they subtend equal angles; required the distance of the observer from the base, the height of the eye being five feet.*



Let $AB = a$, the height of the pedestal;
$BC = b$, the height of the statue;
$DE = c$, the height of the eye from ground;

and $DA = EF = x$, the distance sought. Then $EC^2 = EF^2 + CF^2 = x^2 + (a+b-c)^2$, and

$$EA^2 = EF^2 + ED^2 = x^2 + c^2.$$

But, since the angle $CEB = BEA$, we have [B. IV, Prop. XIII] $EC : EA :: CB : BA$, or

$EC^2 : EA^2 :: CB^2 : BA^2$; or, in symbols,

$$x^2 + (a+b-c)^2 : x^2 + c^2 :: b^2 : a^2.$$

From this proportion, we readily deduce

$$x = \pm\sqrt{\left(\frac{a(a-c)^2 + b(a^2-c^2)}{b-a}\right)};$$

the double sign merely indicating that this value of $x$ may be measured either way, from A towards D, or from D towards A.

**PROBLEM IV.** *Given the three sides* a, b, c *of a triangle, to find:*

1. *The three perpendiculars from the angles upon the opposite sides;*
2. *The area of the triangle;*
3. *The radius of the circumscribed circle;*
4. *The radius of the inscribed circle;*
5. *The radii of the escribed circles* (See Art. 87).

Let ABC be the triangle; and let $a$, $b$, $c$ denote the sides opposite the angles A, B, C respectively, and $P_1$, $P_2$, $P_3$ the perpendiculars drawn from the angles A, B, C; $\Delta$ the area of the triangle; R the radius of the circumscribing circle; $r$ that of the inscribed circle; and $r_1$, $r_2$, $r_3$ the radii of the three escribed circles, which touch the sides $a$, $b$, $c$ externally.

An *escribed* circle has already been defined (Art. 87) as a circle which touches one of the sides of a triangle exteriorly, and the other two sides produced.

We have already found (51) the perpendiculars to be

$$P_1 = \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{2a}, \quad [1]$$

$$P_2 = \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{2b}, \quad [2]$$

$$P_3 = \frac{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{2c}. \quad [3]$$

We have, also, under the same article, found the area to be

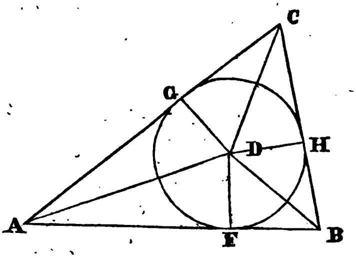
$$\Delta = \left\{ \left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right) \right\}^{\frac{1}{2}}. \quad [4]$$

We know [B. IV, Prop. xxvii] that the diameter of the circumscribing circle, multiplied into either perpendicular, is equal to the product of the sides containing the angle from which the perpendicular is drawn. Hence

$$2\,R \times P_1 = bc, \text{ and } R = \frac{bc}{2P_1}.$$

Substituting for $P_1$ its value already found, we have

$$R = \frac{abc}{\sqrt{(a+b+c)\,(-a+b+c)\,(a-b+c)\,(a+b-c)}}. \quad [5]$$



The sum of the areas of the three triangles ADB, BDC, CDA equals the area ABC. But

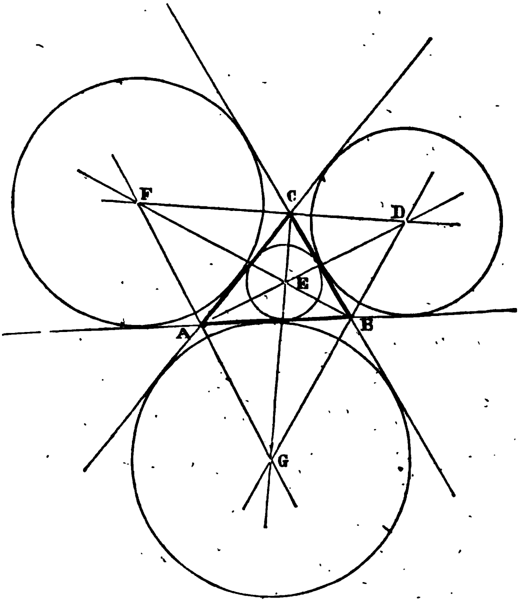
$$ADB = \tfrac{1}{2}rc,$$
$$BDC = \tfrac{1}{2}ra,$$
$$CDA = \tfrac{1}{2}rb;$$

hence

$$\frac{a+b+c}{2} \times r = \Delta, \text{ and} \quad [6]$$

$$r = \frac{2\,\Delta}{a+b+c} = \left\{ \frac{(-a+b+c)\,(a-b+c)\,(a+b-c)}{4\,(a+b+c)} \right\}^{\frac{1}{2}}. \quad [7]$$

We will now seek the radius $r_1$ of the escribed circle, whose centre is at D.



Area DBA + DCA − DBC is evidently equal to the area ABC. Area DBA is equal to the base AB multiplied by half the perpendicular drawn from D upon AB produced : hence area DBA $= c \times \frac{1}{2} r_1 = \frac{1}{2} cr_1$. In a similar way, we find area DCA $= \frac{1}{2} br_1$, and area DBC $= \frac{1}{2} ar_1$. Therefore we have $\frac{b+c-a}{2} \times r_1 = \Delta$, and [8]

$$r_1 = \frac{2\Delta}{b+c-a} = \left\{ \frac{(a+b+c)(a-b+c)(a+b-c)}{4(-a+b+c)} \right\}^{\frac{1}{2}}. \quad [9]$$

By simply permuting, we obtain

$$r_2 = \left\{\frac{(a+b+c)\,(-a+b+c)\,(a+b-c)}{4\,(a-b+c)}\right\}^{\frac{1}{2}}, \qquad [10]$$

$$r_3 = \left\{\frac{(a+b+c)\,(-a+b+c)\,(a-b+c)}{4\,(a+b-c)}\right\}^{\frac{1}{2}}. \qquad [11]$$

Equation [8] readily gives $\frac{1}{r_1} = \frac{-a+b+c}{2\triangle}$; and, in a similar manner, $\frac{1}{r_2} = \frac{a-b+c}{2\triangle}$, and $\frac{1}{r_3} = \frac{a+b-c}{2\triangle}$; therefore $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r^3} = \frac{a+b+c}{2\triangle}$. But we have already from equation [7], $r = \frac{2\triangle}{a+b+c}$, or $\frac{1}{r} = \frac{a+b+c}{2\triangle}$; therefore we have $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$. [12]

Since any side of a triangle, multiplied by the perpendicular which meets it from the opposite angle, gives double the area of the triangle, we have $2\triangle = a\mathrm{P}_1$, or

$$\frac{2\triangle}{\mathrm{P}_1} = a. \qquad [13]$$

In a similar manner, we have

$$\frac{2\triangle}{\mathrm{P}_2} = b, \qquad [14]$$

$$\frac{2\triangle}{\mathrm{P}_3} = c. \qquad [15]$$

Taking the product of [13], [14] and [15], we have

$$\frac{8\triangle^3}{\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3} = abc. \qquad [16]$$

Again, we have
$$2\,\mathrm{R}\,\mathrm{P}_1 = bc, \qquad [17]$$
$$2\,\mathrm{R}\,\mathrm{P}_2 = ac, \qquad [18]$$
$$2\,\mathrm{R}\,\mathrm{P}_3 = ab. \qquad [19]$$

Taking the product of [17], [18] and [19], we have
$$8\,\mathrm{R}^3\,\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3 = a^2b^2c^2. \qquad [20]$$

Extracting the cube root of the product of [16] & [20], we find
$$4\,\mathrm{R}\,\Delta = abc. \qquad [21]$$

Dividing [20] by the square of [16], we find
$$\frac{\mathrm{R}^3\,\mathrm{P}_1^3\,\mathrm{P}_2^3\,\mathrm{P}_3^3}{8\,\Delta^6} = 1, \text{ or } \mathrm{R}\,\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3 = 2\,\Delta^2. \qquad [22]$$

By taking the continued product of [7], [9], [10] and [11], we have
$$r\,r_1r_2r_3 = \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{16} = \Delta^2. \qquad [23]$$

By multiplying [5] and [7] together, or [7] and [21], we have
$$2\,\mathrm{R}r = \frac{abc}{a+b+c}. \qquad [24]$$

By a similar multiplication, we find
$$\left.\begin{aligned} 2\,\mathrm{R}r_1 &= \frac{abc}{-a+b+c}, \\ 2\,\mathrm{R}r_2 &= \frac{abc}{a-b+c}, \\ 2\,\mathrm{R}r_3 &= \frac{abc}{a+b-c}. \end{aligned}\right\} \qquad [25]$$

By combining the values of [9], [10] and [11], by two and two, we find
$$r_1r_2+r_1r_3+r_2r_3 = \left(\frac{a+b+c}{2}\right)^2. \qquad [26]$$

We may also deduce

$$\frac{1}{P_1}+\frac{1}{P_2}+\frac{1}{P_3}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}. \qquad [27]$$

$$\tfrac{1}{2}Rr=\frac{\Delta^2}{P_1P_2+P_2P_3+P_3P_1}. \qquad [28]$$

$$\left.\begin{aligned} P_1 &= \frac{2\,r_2r_3}{r_2+r_3},\\ P_2 &= \frac{2\,r_1r_3}{r_1+r_3},\\ P_3 &= \frac{2\,r_1r_2}{r^1+r^2}. \end{aligned}\right\} \qquad [29]$$

$$\left.\begin{aligned} \frac{1}{P_1}+\frac{1}{P_2} &= \frac{1}{P_3}+\frac{1}{r_3},\\ \frac{1}{P_1}+\frac{1}{P_3} &= \frac{1}{P_2}+\frac{1}{r_2},\\ \frac{1}{P_2}+\frac{1}{P_3} &= \frac{1}{P_1}+\frac{1}{r_1}. \end{aligned}\right\} \qquad [30]$$

$$r_1+r_2+r_3=4R+r. \qquad [31]$$

Any of the foregoing expressions, when properly translated into common language, leads to a theorem. We will translate some of the most interesting ones.

Equation [12] gives the following

**Theorem.** *The reciprocal of the radius of the inscribed circle is equal to the sum of the reciprocals of the radii of the three escribed circles.*

Equation [21] yields the following

THEOREM. *Four times the radius of the circumscribed circle, into the area of the triangle, is equal to the continued product of the three sides.*

Equation [22] gives this

THEOREM. *The radius of the circumscribed circle, into the continued product of the three perpendiculars, is equal to twice the square of the area.*

Equation [23] gives this

THEOREM. *The radius of the inscribed circle, into the continued product of the radii of the three escribed circles, is equal to the square of the area.*

Equation [27] gives this

THEOREM. *The sum of the reciprocals of the three perpendiculars is equal to the sum of the reciprocals of the three radii of the escribed circles.*

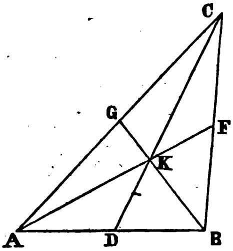
By combining the equations already formed, new ones would arise, which might still afford interest. Thus, by a comparison of equations [22] and [23], we find

$$R\,P_1P_2P_3 = 2\,r\,r_1r_2r_3; \qquad [32]$$

which gives this

THEOREM. *The radius of the circumscribed circle, into the continued product of the three perpendiculars, is equal to the diameter of the inscribed circle, into the continued product of the three radii of the escribed circles.*

**Problem V.** *Given the three sides of a triangle, to find the three lines drawn from the angles to the middle points of the opposite sides.*



Let ABC be the triangle. Denote the sides opposite the angles A, B, C, by $a$, $b$, $c$ respectively; also denote the lines drawn from the angles A, B, C to the middle points of the opposite sides, by $m_1$, $m_2$, $m_3$ respectively. Then (B. II, Prop. XII) we shall have

$$2\,\mathrm{AF}^2 + 2\,\mathrm{BF}^2 = \mathrm{AB}^2 + \mathrm{AC}^2, \text{ or}$$

$$2\,m_1^2 + \tfrac{1}{2}a^2 = c^2 + b^2; \text{ which gives}$$

$$4\,m_1^2 = -a^2 + 2\,b^2 + 2\,c^2. \qquad [1]$$

In a similar manner, we find

$$4\,m_2^2 = -b^2 + 2\,c^2 + 2\,a^2, \qquad [2]$$

$$4\,m_3^2 = -c^2 + 2\,a^2 + 2\,b^2. \qquad [3]$$

Equations [1], [2], and [3], readily give

$$\left.\begin{array}{l} m_1 = \tfrac{1}{2}\sqrt{-a^2 + 2\,b^2 + 2\,c^2}, \\ m_2 = \tfrac{1}{2}\sqrt{-b^2 + 2\,c^2 + 2\,a^2}, \\ m_3 = \tfrac{1}{2}\sqrt{-c^2 + 2\,a^2 + 2\,b^2}. \end{array}\right\} \qquad [4]$$

We will now deduce a few remarkable relations, which, when properly translated, will give some beautiful theorems.

Taking the sum of [1], [2] and [3], we obtain

$$4\,(m_1^2+m_2^2+m_3^2) = 3\,(a^2+b^2+c^2). \qquad [5]$$

If we take the product of [1] and [2], we shall have

$$16\,m_1^2m_2^2 = -2a^4+5a^2b^2+2a^2c^2-2b^4+2b^2c^2+4c^4. \quad [6]$$

By simply permuting, we find

$$16\,m_2^2m_3^2 = -2b^4+5b^2c^2+2b^2a^2-2c^4+2c^2a^2+4a^4, \quad [7]$$

$$16\,m_3^2m_1^2 = -2c^4+5c^2a^2+2c^2b^2-2a^4+2a^2b^2+4b^4. \quad [8]$$

Taking the sum of [6], [7] and [8], we obtain

$$16\,(m_1^2m_2^2+m_2^2m_3^2+m_3^2m_1^2) = 9\,(a^2b^2+b^2c^2+c^2a^2). \quad [9]$$

If, from the square of [5], we subtract twice [9], we shall have

$$16\,(m_1^4+m_2^4+m_3^4) = 9\,(a^4+b^4+c^4). \qquad [10]$$

The point where these lines trisect each other, is the centre of gravity of the triangle (B. IV, Prop. XIV, and Art. 73). If we denote the distances AK, BK, CK, by $d_1$, $d_2$, $d_3$ respectively, we shall have $m_1 = \frac{3}{2}\,d_1$, $m_2 = \frac{3}{2}\,d_2$, $m_3 = \frac{3}{2}\,d_3$. These values substituted in [5], [9] and [10], cause them to become

$$3(d_1^2+d_2^2+d_3^2) = a^2+b^2+c^2, \qquad [11]$$

$$9(d_1^2d_2^2+d_2^2d_3^2+d_3^2d_1^2) = a^2b^2+b^2c^2+c^2a^2, \qquad [12]$$

$$9(d_1^4+d_2^4+d_3^4) = a^4+b^4+c^4. \qquad [13]$$

Equation [5] is equivalent to the following

THEOREM. *Four times the sum of the squares of the lines drawn from the angles of a triangle to the middle points of the opposite sides, is equal to three times the sum of the squares of the sides.*

Equation [9] gives this

THEOREM. *Sixteen times the sum of the products, taken two at a time, of the squares of the lines drawn from the angles of a triangle to the middle points of the opposite sides, is equal to nine times the sum of the products, taken two at a time, of the squares of the sides.*

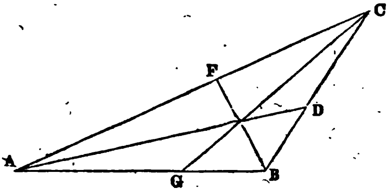
Equation [10] gives this

THEOREM. *Sixteen times the sum of the fourth powers of the lines drawn from the angles of a triangle to the middle points of the opposite sides, is equal to nine times the sum of the fourth powers of the sides.*

Equations [11], [12] and [13] would lead to beautiful theorems in reference to the distances of the centre of gravity of three equal bodies, and the mutual distances of the bodies themselves. For it is evident that, instead of considering the triangle as material, we may suppose equal weights placed at its vertices; and, under this point of view, equation [11] will give the following

THEOREM. *Three times the sum of the squares of the distances of three equal bodies from their common centre of gravity, is equal to the sum of the squares of their mutual distances.*

**Problem VI.** *Given the three sides of a triangle, to find the lines bisecting the angles, and terminating in the opposite sides.*



Let the sides opposite the angles A, B, C be denoted by $a$, $b$, $c$ respectively; also denote the lines which bisect the angles A, B, C respectively by $l_1$, $l_2$, $l_3$. Then (B. IV, Prop. XIII) we shall have

$$AB : AC :: BD : DC.$$ Consequently

$$AB + AC : AB :: BD + DC : BD,$$

$$AB + AC : AC :: BD + DC : DC;$$ or, in symbols,

$$c + b : c :: a : BD,$$

$$c + b : b :: a : DC.$$

Hence $BD = \frac{ac}{c+b}$; $DC = \frac{ab}{c+b}$.

Again (B. IV, Prop. XXVI), we have

$$AB \times AC = AD^2 + BD \times DC;$$ which becomes

$$cb = l_1^2 + \frac{a^2bc}{(c+b)^2}.$$ This readily gives

$$l_1^2 = cb - \frac{a^2bc}{(c+b)^2},$$ or

$$l_1^2 = \frac{cb(a+b+c)(-a+b+c)}{(c+b)^2}. \quad [1]$$

In a similar manner, we find

$$l_2^2 = \frac{ac(a+b+c)(a-b+c)}{(a+c)^2}, \qquad [2]$$

$$l_3^2 = \frac{ba(a+b+c)(a+b-c)}{(b+a)^2}. \qquad [3]$$

Equations [1], [2], [3], give

$$\left.\begin{aligned} l_1 &= \frac{\sqrt{cb(a+b+c)(-a+b+c)}}{c+b}, \\ l_2 &= \frac{\sqrt{ac(a+b+c)(a-b+c)}}{a+c}, \\ l_3 &= \frac{\sqrt{ba(a+b+c)(a+b-c)}}{b+a}. \end{aligned}\right\} \qquad [4]$$

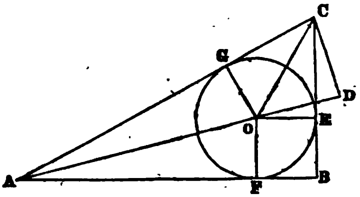
Taking the continued product of these values given by [4], we find

$$l_1l_2l_3 = \frac{abc(a+b+c)\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}{(a+b)(b+c)(c+a)}.$$

The expression $\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$ is equal to four times the area of the triangle (see Prob. 4), which we will denote by 4 $\triangle$; so that we shall have

$$l_1l_2l_3 = \frac{4abc(a+b+c)\triangle}{(a+b)(b+c)(c+a)}. \qquad [5]$$

**Problem** VII. *To determine a right-angled triangle, having given the hypothenuse and difference of two lines drawn from the two acute angles to the centre of the inscribed circle.*



Let

$AC = h;$

$AO = x + d;$

$CO = x - d,$

$d$ being half the difference of the lines AO and CO. Produce AO, and draw CD perpendicular to AO thus produced. Then since the angle COD is equal to the sum of CAO and ACO (B. I, Prop. xxv), and since CAO is half the angle CAB, and ACO the half of ACB, it follows that COD is half a right angle; consequently OCD is also half a right angle: therefore OD is to OC, as the side of a square is to its diagonal; that is, as 1 to $\sqrt{2}$ [B. IV, Art. 64]. Hence

$$OD = CD = \frac{x-d}{\sqrt{2}}, \text{ and } AD = x + d + \frac{x-d}{\sqrt{2}}.$$

Again, $AC^2 = AD^2 + CD^2$; or, which is in symbols

$$h^2 = \left(x + d + \frac{x-d}{\sqrt{2}}\right)^2 + \left(\frac{x-d}{\sqrt{2}}\right)^2. \qquad [1]$$

This readily gives $x = \left(\frac{h^2-(2-\sqrt{2})d^2}{2+\sqrt{2}}\right)^{\frac{1}{2}}.$ [2]

Having found $x$, we of course know AO and CO. Let AO be denoted by $a$, and CO by $b$; also let the radius of the inscribed circle be denoted by $r$.

$$\text{Then } AF = \sqrt{AO^2 - OF^2} = \sqrt{a^2 - r^2},$$

$$CE = \sqrt{CO^2 - OE^2} = \sqrt{b^2 - r^2}.$$

But AF = AG, and CE = CG; therefore

$$\sqrt{a^2 - r^2} + \sqrt{b^2 - r^2} = h. \qquad [3]$$

This readily gives

$$r = \left\{ b^2 - \left( \frac{h^2 + b^2 - a^2}{2h} \right)^2 \right\}^{\frac{1}{2}}, \text{ or}$$

$$r = \frac{\sqrt{(a+b+h)(-a+b+h)(a-b+h)(a+b-h)}}{2h}. \qquad [4]$$

This value of $r$ might have been found as follows: In the triangle AOC, all the sides are known; and it is required to find the perpendicular OG, which is $r$, the radius of the circle. Under Prob. 4, we have found the perpendiculars when the three sides are known. The perpendicular OG found in this way, will be precisely the same as the above value of $r$.

Now, having found $r$, we will denote AB by $x$, and CB by $y$; then AF $= x - r$, and CE $= y - r$. Hence,

$$\text{AF} + \text{CE} = h = x + y - 2r. \qquad [5]$$

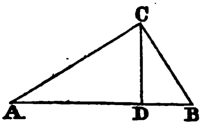
Again, $xy$ = double the area of the triangle ABC; also $(x+y+h)r$ = double the area. Therefore,

$$xy = (x + y + h)r. \qquad [6]$$

Equations [5] and [6] readily make known $x$ and $y$, or the sides of the triangle: these values are

$$\left. \begin{aligned} x &= \frac{h + 2r + \sqrt{h^2 - 4hr - 4r^2}}{2}, \\ y &= \frac{h + 2r - \sqrt{h^2 - 4hr - 4r^2}}{2}. \end{aligned} \right\} \qquad [7]$$

**Problem VIII.** *In a right-angled triangle, having given the perimeter, and the perpendicular drawn from the right angle upon the hypothenuse; to find the sides of the triangle.*



Let $S =$ sum of sides, or perimeter;
$P =$ perpendicular CD;
$AC = x$, and $BC = y$.

Then will $\sqrt{x^2+y^2} = AB$.

$xy =$ double the area of triangle ABC; also,

$P\sqrt{x^2+y^2} =$ double the area of triangle ABC:

therefore $$xy = P\sqrt{x^2+y^2}. \qquad [1]$$

We also have $$x + y + \sqrt{x^2+y^2} = S. \qquad [2]$$

Equation [2] becomes, by transposing and squaring,

$$(x+y)^2 = (S - \sqrt{x^2+y^2})^2, \text{ or}$$

$$x^2 + 2xy + y^2 = S^2 - 2S\sqrt{x^2+y^2} + x^2 + y^2, \text{ or}$$

$$2xy = S^2 - 2S\sqrt{x^2+y^2}. \qquad [3]$$

Taking the double of [1], we have

$$2xy = 2P\sqrt{x^2+y^2}. \qquad [4]$$

Equating right-hand members of [3] and [4], we have

$$2P\sqrt{x^2+y^2} = S^2 - 2S\sqrt{x^2+y^2}, \text{ or}$$

$$\sqrt{x^2+y^2} = \frac{S^2}{2(P+S)}. \qquad [5]$$

Using this value in [1], we find

$$xy = \frac{PS^2}{2(P+S)}. \qquad [6]$$

Squaring [5], we have

$$x^2+y^2 = \frac{S^4}{4(P+S)^2}. \qquad [7]$$

Adding twice [6] to [7], and also subtracting twice [6] from [7], we have

$$x^2+2xy+y^2 = \frac{S^2(S^2+4PS+4P^2)}{4(P+S)^2}, \qquad [8]$$

$$x^2-2xy+y^2 = \frac{S^2(S^2-4PS-4P^2)}{4(P+S)^2}. \qquad [9]$$

Extracting the square root of [8] and [9], we have

$$x+y = \frac{S(S+2P)}{2(P+S)}, \qquad [10]$$

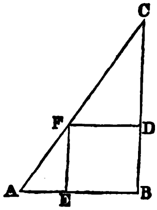
$$x-y = \frac{S\sqrt{S^2-4PS-4P^2}}{2(P+S)}. \qquad [11]$$

Hence

$$x = \frac{S(S+2P)+S\sqrt{S^2-4PS-4P^2}}{4(P+S)}, \qquad [12]$$

$$y = \frac{S(S-2P)-S\sqrt{S^2-4PS-4P^2}}{4(P+S)}. \qquad [13]$$

**Problem IX.** *To determine a right-angled triangle; having given the hypothenuse, and side of the inscribed square.*



Let $h$ = hypothenuse;

$s$ = side of inscribed square;

AB = $x$, and BC = $y$.

Then AE : AB : : EF : BC; that is,

$x-s : x :: s : y$, or

$$xy = (x+y)s. \qquad [1]$$

Again, $x^2+y^2 = h^2$. [2]

Add twice [1] to [2], and we obtain

$$x^2 + 2xy + y^2 = 2s(x+y) + h^2, \text{ or}$$

$$(x+y)^2 - 2s(x+y) = h^2, \qquad [3]$$

which is a quadratic in terms of $x+y$ : hence we find

$$x+y = s + \sqrt{s^2+h^2}. \qquad [4]$$

Substituting this value of $x+y$ in [1], and we shall obtain $xy = s^2 + s\sqrt{s^2+h^2}$. [5]

From the square of [4] subtracting four times [5], we obtain $x^2 - 2xy + y^2 = h^2 - 2s^2 - 2s\sqrt{s^2+h^2}$. [6]

Extracting the square root of [6], we have

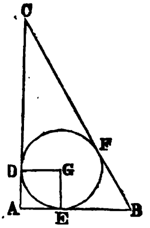
$$x-y = \sqrt{h^2 - 2s^2 - 2s\sqrt{s^2+h^2}}. \qquad [7]$$

Taking half the sum of [4] and [7], and also half their difference, we obtain

$$x = \frac{s + \sqrt{s^2+h^2} + (h^2 - 2s^2 - 2s\sqrt{s^2+h^2})^{\frac{1}{2}}}{2}, \qquad [8]$$

$$y = \frac{s + \sqrt{s^2+h^2} - (h^2 - 2s^2 - 2s\sqrt{s^2+h^2})^{\frac{1}{2}}}{2}. \qquad [9]$$

**PROBLEM X.** *To determine a right-angled triangle; having given the hypothenuse, and the radius of the inscribed circle.*



Let $h$ = hypothenuse;
$r$ = radius of the inscribed circle;
AB = $x$, and AC = $y$.

Then BF = BE = $x-r$,
and CF = CD = $y-r$; therefore
BC = BF + CF = $x+y-2r = h$, or

$$x+y = 2r+h. \qquad [1]$$

Again, $xy$ = double the area;
$(x+y+h)r = (x+y)r + hr$ = double the area :
therefore $xy = (x+y)r + hr$. [2]

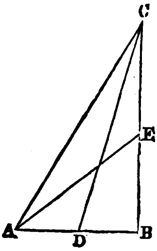
In [2], for $x+y$ substitute its value given by [1], and it will become $xy = 2r(r+h)$. [3]

Equations [1] and [3] readily give

$$x = \frac{2r+h+\sqrt{h^2-4hr-4r^2}}{2}, \qquad [4]$$

$$y = \frac{2r+h-\sqrt{h^2-4hr-4r^2}}{2}. \qquad [5]$$

**Problem XI.** *To determine a right-angled triangle; having given the lengths of the lines drawn from the acute angles, to the middle points of the opposite sides.*



Let AE $= a$, and CD $= b$;
also AB $= x$, and BC $= y$.
Then will $AB^2 + BE^2 = a^2$, or

$$x^2 + \tfrac{1}{4}y^2 = a^2. \qquad [1]$$

In a similar way, we find

$$y^2 + \tfrac{1}{4}x^2 = b^2. \qquad [2]$$

Equations [1] and [2], when cleared of fractions, become

$$4x^2+y^2 = 4a^2, \qquad [3]$$

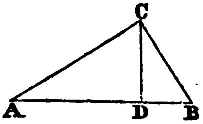
$$x^2+4y^2 = 4b^2. \qquad [4]$$

These equations give

$$x = 2\sqrt{\frac{4a^2-b^2}{15}}, \qquad [5]$$

$$y = 2\sqrt{\frac{4b^2-a^2}{15}}. \qquad [6]$$

**Problem** XII. *To determine a triangle; having given the base, the perpendicular, and the difference of the two other sides.*



Let $p =$ the perpendicular,
$d =$ half the difference of sides;
$x+d =$ AC,
$x-d =$ BC.
$b =$ half the base;
$b+y =$ AD,
$b-y =$ DB.

Then from the right-angled triangles ADC, BDC, we have

$$(b+y)^2+p^2 = (x+d)^2, \qquad [1]$$

$$(b-y)^2+p^2 = (x-d)^2. \qquad [2]$$

Expanding [1] and [2], and then taking half their sum and one-fourth of their difference, we obtain

$$b^2+y^2+p^2 = x^2+d^2, \qquad [3]$$

$$by = dx. \qquad [4]$$

These equations readily give

$$x = b\left(\frac{b^2-d^2+p^2}{b^2-d^2}\right)^{\frac{1}{2}}, \qquad [5]$$

$$y = d\left(\frac{b^2-d^2+p^2}{b^2-d^2}\right)^{\frac{1}{2}}. \qquad [6]$$

Hence $$\text{AC} = b\left(\frac{b^2-d^2+p^2}{b^2-d^2}\right)^{\frac{1}{2}}+d, \qquad [7]$$

$$\text{BC} = b\left(\frac{b^2-d^2+p^2}{b^2-d^2}\right)^{\frac{1}{2}}-d. \qquad [8]$$

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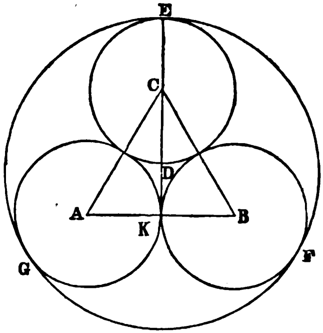
**PROBLEM XIII.** *To determine the radii of three equal circles, described in a given circle, to touch each other, and also the circumference of the given circle.*



Let D be the centre of the given circle, whose radius we will denote by R; also let A, B, C be the centres of the three equal circles, whose common radius we will denote by $r$. Then joining A, B and C, we have the triangle ABC equilateral, each of whose sides is $2r$. Drawing CK perpendicular to AB, the right-angled triangle AKC gives

$$CK^2 = AC^2 - AK^2, \text{ or in symbols}$$

$$CK^2 = 4r^2 - r^2 = 3r^2; \text{ consequently,}$$

$$CK = r\sqrt{3}. \qquad [1]$$

Now [B. IV, Prop. XIV] $CD = \frac{2}{3}CK = \frac{2}{3}r\sqrt{3}$. [2]

But $ED = EC + CD$; that is,

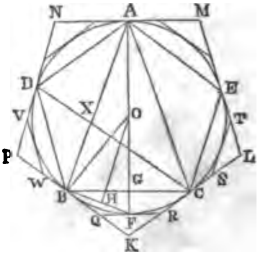
$$R = r + \tfrac{2}{3}r\sqrt{3}: \qquad [3]$$

from which we readily find

$$r = \frac{3R}{3+2\sqrt{3}} = \frac{R}{1+2\sqrt{\frac{1}{3}}}. \qquad [4]$$

**Problem XIV.** *Given the radius* r *of a circle, to find the sides of the inscribed and circumscribed pentagons and decagons.*

1. *The inscribed pentagon.* Let ADBCE be the pentagon inscribed in the circle (B. V, Prop. viii), and let O be the centre of the circumscribing circle. Join AB, AC, and draw AF perpendicular to BC; then, by known properties, BC is bisected in G, and the line AF passes through the centre O of the circle. Since the angle ADX is measured by half the arc AEC (B. III, Prop. viii), and the angle AXD is measured by half the sum of the arcs AD and BC (B. III, Prop. xiii), it follows that these angles are equal, and the triangle DAX is isosceles.



Again, the angles DAX and BAC are equal, being measured by halves of the equal arcs DB, BC : hence the triangles ABC and DAX are similar. But the triangle DAX is obviously equal to BCX; therefore

$$AB : BC :: BC : BX;$$

but $BX = AB - AX = AB - BC$; therefore

$$AB : BC :: BC : AB - BC. \qquad [1]$$

Put $BC = 2x$, or $BG = x$; $BA = y$, and $BF = z$; and the above proportion will become

$$y : 2x :: 2x : y - 2x; \text{ which gives}$$

$$y = (1 + \sqrt{5})x. \qquad [2]$$

Now we have $OG = \sqrt{r^2 - x^2}$, $AG = \sqrt{y^2 - x^2}$; and hence $AG = AO + OG$ gives $\sqrt{y^2 - x^2} = r + \sqrt{r^2 - x^2}$, or, squaring, $y^2 - 2\,r^2 = 2\,r\sqrt{r^2 - x^2}$. Again squaring, we have

$$y^4 - 4\,r^2y^2 + 4\,r^2x^2 = 0. \qquad [3]$$

Substituting the value of $y$ as given by [2], we find the side of an inscribed pentagon

$$2\,x = \tfrac{1}{2}\,r\sqrt{10 - 2\sqrt{5}}. \qquad [4]$$

We have $y = x(1 + \sqrt{5}) = \tfrac{1}{2}\,r\sqrt{10 + 2\sqrt{5}}$, [5]

$OG = \sqrt{r^2 - x^2} = \sqrt{r^2 - \tfrac{1}{8}(5 - \sqrt{5})r^2} = \tfrac{1}{4}\,r(1 + \sqrt{5})$. [6]

2. *The inscribed decagon.* Join BF : then since AF bisects the line BC at right angles, it bisects the arc BFC in F ; and hence BF is the side of the inscribed decagon. But ABF being a right angle, since it is in a semicircle, we have $BF^2 = FA^2 - AB^2$, or, in symbols,

$$z^2 = 4\,r^2 - y^2 = 4\,r^2 - \tfrac{1}{4}(10 + 2\sqrt{5})r^2 = \tfrac{1}{2}(3 - \sqrt{5})r^2\,;$$

or, extracting the square root, we have for the side of the inscribed decagon $z = \tfrac{1}{2}\,r(\sqrt{5} - 1)$. [7]

3. *The circumscribing pentagon.* The inscribed and circumscribed pentagons being regular, are similar figures, and their sides are as the perpendiculars from the centre upon the sides. That is, if PK be a side of the circumscribing pentagon, we shall have

$OG : OB :: BC : PK$; or, in symbols,

$$PK = \frac{OB \times BC}{OG} = \frac{r \cdot \tfrac{1}{2}\,r\sqrt{10 - 2\sqrt{5}}}{\tfrac{1}{4}\,r(1 + \sqrt{5})} = 2\,r\sqrt{5 - 2\sqrt{5}}. \qquad [8]$$

4. *The circumscribing decagon.* Let QR be one of the sides; and draw OH perpendicular to BF, which it bisects in H. Also by similar triangles ABF, OHF, we have $OH = \frac{1}{2} AB = \frac{1}{2} y = \frac{1}{4} r\sqrt{10+2\sqrt{5}}$.

Also, as in the last case,

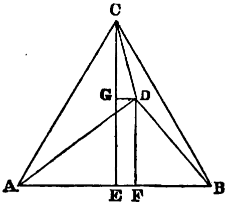
$$OH : OF :: BF : QR; \text{ which gives}$$

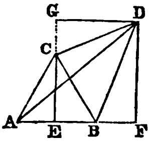
$$QR = \frac{OF \times BF}{OH} = \frac{r . \frac{1}{2} r(\sqrt{5}-1)}{\frac{1}{4} r\sqrt{10+2\sqrt{5}}} = 2r\sqrt{\frac{5-2\sqrt{5}}{5}}.$$

**Problem XV.** *Given the lengths of three lines drawn from a point to the three angles of an equilateral triangle, to find its side.*

Let ABC be the triangle, and D the point.

Put $AD = a$,
$BD = b$,
$CD = c$;
$AB = 2x$,
$EF = y$,
$FD = z$.





Then will $AF = x + y$, and $BF = x - y$ or $y - x$.

$$CE = \sqrt{AC^2 - AE^2} = \sqrt{4x^2 - x^2} = x\sqrt{3};$$

and $CG = x\sqrt{3} - z$, or $z - x\sqrt{3}$.

Hence we have the following relations true, whether the point D is *within* the equilateral triangle, or *without* it.

$$(x+y)^2 + z^2 = a^2, \qquad [1]$$

$$(x-y)^2 + z^2 = b^2, \qquad [2]$$

$$(x\sqrt{3}-z)^2 + y^2 = c^2. \qquad [3]$$

Subtracting [2] from [1], we find

$$4\,xy = a^2 - b^2, \text{ or } y = \frac{a^2-b^2}{4x}; \qquad [4]$$

consequently $$y^2 = \frac{(a^2-b^2)^2}{16x^2}. \qquad [5]$$

Equation [1] gives immediately $z = \sqrt{a^2-(x+y)^2}$; in which, substituting for $y$ its value given by [4], we have

$$z = \sqrt{a^2 - \left(x + \frac{a^2-b^2}{4x}\right)^2}. \qquad [6]$$

This value of $z$, and the value of $y^2$ given by [5], being substituted in [3], will cause it to become

$$\left\{x\sqrt{3} - \sqrt{a^2 - \left(x + \frac{a^2-b}{4x}\right)^2}\right\}^2 + \frac{(a^2-b^2)^2}{16x^2} = c^2. \quad [7]$$

This equation contains only the unknown $x$, which value may therefore be found : the reduction leads to

$$16x^4 - 4(a^2+b^2+c^2)x^2 = -a^4-b^4-c^4+a^2b^2+b^2c^2+c^2a^2. \quad [8]$$

This equation, when solved by the rule for quadratics, gives

$$2\,x = \left\{\frac{a^2+b^2+c^2 \pm \sqrt{6(a^2b^2+b^2c^2+c^2a^2)-3(a^4+b^4+c^4)}}{2}\right\}^{\frac{1}{2}}. [9]$$

We must use the + sign when the point is within the triangle, as in the first figure; and the − sign when the point is without, as in the second figure.

If we suppose a triangle to be formed with the three lines $a$, $b$, $c$, and denote its area by $\Delta$, expression [9] will become

$$2\,x = \left(\frac{a^2+b^2+c^2 \pm 4\Delta\sqrt{3}}{2}\right)^{\frac{1}{2}}. \qquad [10]$$

By referring to Art. 28, it will be seen that we have already given a geometrical solution of this problem in the case where the point is within the triangle, which corresponds with the above expression when the + sign is used. A geometrical solution for the case where the point is without the triangle, will be as follows :

Let the equilateral triangle HFG (figure of Art. 28) be constructed on the other side of the line GF ; then, by joining the vertex H and D, it will give the side of the triangle required.